A Probability Paradox

by Harsh Patil - Tuesday, December 15, 2020

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Imagine a casino with an entry fee of INR X. Inside there is only one game.

The dealer tosses a coin until it lands on a tails. He gives you INR $latex 2^n$ if it takes $latex n$ tosses for the first tails to appear. So if on the first toss the coin the coin lands tails you win INR 2, if the first toss is a heads and the second tails you win INR 4, if the first two tosses land on heads and the third one on tails you win INR 8 and so on. You can play the game only once. Potentially you can win a reward as big as you can imagine. But remember that you had initially paid INR X as an entry fee, so your net winnings will beINR $latex (2^n-X)$ if the first tails appears on the n-th toss. We would like to know how much do we stand to win (or lose) in such a game on average.

For this we calculate the expectation of the payoff (net profit/loss). The coin lands on tails on the first toss with probability $latex \frac{1}{2}$. The probability that the first toss is heads and the second toss is tails is $latex \frac{1}{4}$. The probability that the first tails appears after n tosses is $latex \frac{1}{2^n}$.
So our expectation turns out be
$latex -X+\frac{1}{2}(2)+\frac{1}{4}(4)+\frac{1}{8}(8)\dots $.
This is equal to $latex =-X+1+1+1+1\dots=\infty$! Thus the expectation is positive no matter how big X is. So no matter how much money the casino charges for the entry, a 'rational' player will always chose to play the game. Even if the entry fee is a million rupees!

Despite this most people would be unwilling to part with a million rupees to play this game once. One has to agree that this is reasonable as there is a 50% chance that you will only win back INR 2. Does this mean that the expectation is not a good way to find out if a game is profitable or not? If so then what is a good way of deciding the appropriate value of X?

This problem was posed by the Swiss mathematician Nicholas Bernoulli in a letter to his friend **Pierre Montmort**. His brother **Daniel Bernoulli** was the first to attempt a resolution to this paradox. This paradox is also called the 'St. Petersburg Paradox' as he once lived in St. Petersburg. He based his solution on the assumption that the significance of gain (or loss) of a certain amount of money depends on how wealthy the person is.

> The determination of the value of an item must not be based on the price, but rather on the utility it yields. There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount.
>
> Daniel Bernoulli

He took into account the player's total wealth. Earlier it did not matter how wealthy the player is. To capture this mathematically he used the logarithmic function. Suppose a person has a total wealth of INR w initially and after a certain transaction it becomes INR (w+c) where c could be positive or negative. Then we denote the payoff by $latex ln(w+c)-ln(w)$. Now suppose a person has a total wealth of INR 2 and he gains an additional INR 2 then the payoff is given by $latex ln(4)-ln(2)\approx 0.693$. Now suppose he initially had INR4 and gains INR 2 then his payoff comes out be $latex ln(6)-ln(4)\approx 0.405$. Thus the payoff is lower when one starts out with a larger wealth.

Getting back to our game, suppose a person with wealth w wins INR 2^k then his payoff is given by $latex ln(w+2^k-X)-ln(w)$. So our expectation is given by the following expression
$latex \sum_{k=1}^{\infty}\frac{1}{2^k}(ln(w+2^k-X)-ln(w))$.
For what choice of w and X is the expectation positive?

Let us fix w and calculate X such that the expectation is positive. Suppose a man's total wealth is w=100. Then at around X=7 the above sum becomes positive. So a person with total wealth of INR 100 should be willing to play this game if the entry ticket is INR 7 or less. That seems reasonable compared to our earlier conclusion that a person must be willing to pay any price to play the game. For a millionaire i.e. $latex w=10^6$ the value of expectation is positive for X=20. The choice of logarithmic function was arbitrary and in it's place any concave function could have been taken, for example the square root function. But the story does not end here!

If we modify our game such that the casino now pays INR $latex 2^{2^{k}}$ then we again get an infinite expectation with our new logarithmic payoffs. Since Bernoulli many other ways of resolving this paradox have been proposed. A casino that promises to pay arbitrarily large sums of money as reward should have infinite resources. But the wealth of any actual casino has to be finite. Suppose then that the total wealth of a casino is W. The rules need to modified as the casino cannot pay the player INR $latex 2^k$ if $latex 2^k$ exceeds W i.e k exceeds $latex log_2(W)$. We modify the rules as follows: If the number of tosses is less than or equal to $latex log_2(W)$ then the casino pays INR $latex 2^k$ and if the number of tosses is more than $latex log_2(W)$ the casino pays INR W. Let us examine the expectation of this modified game. Let L be the greatest integer less than or equal to $latex log_2(W)$.
$latex \sum_{k=1}^{L}\frac{1}{2^k}\cdot2^k+\sum_{k=L+1}^{\infty}\frac{1}{2^k}W=L+\frac{W}{2

^{L}}$.
For $latex W=10^6$, L=19 and $latex L+\frac{W}{2^L}=20.91$ even in the case that the casino has a wealth of million rupees the expectation is quite modest. For a casino with a wealth of a billion rupees the expectation is INR 30.86. Thus one could say that the origin of the earlier paradox lies in the assumption that the casino has infinite wealth.