A Problem in Linear Algebra

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In this article we shall show that the characteristic polynomial of both $latex {AB}&fg=000000$ and $latex {BA}&fg=000000$ are the same, where $latex {A}&fg=000000$ and $latex {B}&fg=000000$ are $latex {n\times n}&fg=000000$ matrices over $latex {R}&fg=000000$, a ring with unity.

Let $latex {\Phi_T\in R[\lambda]}&fg=000000$ denote the characteristic polynomial of $latex {T}&fg=000000$. Thus, we intend to show, $latex {\Phi_{AB}=\Phi_{BA}}&fg=000000$.

We first take $latex {R=\mathbb{C}}&fg=000000$ and specialize in this ring. By multiplicity of the determinant function, we have that the characteristic polynomial of two similar matrices are the same.

We keep $latex {A}&fg=000000$ fixed and let $latex {B}&fg=000000$ be a diagonalizable (semisimple) matrix. Let $latex {T}&fg=000000$ be an invertible matrix such that $latex {TBT^{-1}}&fg=000000$ is diagonal. Then,

$latex \displaystyle \Phi_{AB}&fg=000000$

$latex \displaystyle = \Phi_{(TAT^{-1})(TBT^{-1})}&fg=000000$

$latex \displaystyle =det(\lambda I - (TAT^{-1})(TBT^{-1}))&fg=000000$

$latex \displaystyle =det((\lambda I - (TBT^{-1})(TAT^{-1}))^{T})&fg=000000$

$latex \displaystyle =\Phi_{(TBT^{-1})(TAT^{-1})}&fg=000000$

$latex \displaystyle =\Phi_{BA}&fg=000000$

Fact: The set of semisimple matrices is dense in $latex {M_n(\mathbb{C})}&fg=000000$.

By making use of the this and continuity of det function we conclude that

$latex \displaystyle \Phi_{AB}=\Phi_{BA},&fg=000000$

$latex \displaystyle \forall A,B\in M_n(\mathbb{C}).&fg=000000$

Let $latex {X = [x_{ij}]_{n\times n}}&fg=000000$ and $latex {Y=[y_{ij}]_{n\times n}}&fg=000000$ be two matrices where $latex {\{x_{ij}\}}&fg=000000$ and $latex {\{y_{ij}\}}&fg=000000$ are $latex {2n^2}&fg=000000$ variables. Then,

$latex \displaystyle \Phi_{XY}-\Phi_{YX}\in\mathbb{C}[\{x_{ij}\},\{y_{ij}\},\lambda].&fg=000000$

We call this polynomial as $latex {P}&fg=000000$. Corresponding to this polynomial we have a function $latex {\tilde{P} : \mathbb{C}^{2n^2+1}\rightarrow\mathbb{C}}&fg=000000$ where,

$latex \displaystyle \tilde{P}(\{a_{ij}\},\{b_{ij}\},l)=det(lI-[a_{ij}][b_ij])-det(lI-[a_{ij}][b_ij])&fg=000000$

As $latex {\Phi_{AB}=\Phi_{BA}}&fg=000000$, we have that $latex {\tilde{P}}&fg=000000$ is the zero function. Thus, we have that $latex {P}&fg=000000$ is the zero polynomial.

As

$latex \displaystyle \mathbb{Z}[\{x_{ij}\},\{y_{ij}\},\lambda]\subset \mathbb{C}[\{x_{ij}\},\{y_{ij}\},\lambda]&fg=000000$

the identity, $latex {P=0}&fg=000000$ holds for entries in $latex {\mathbb{Z}}&fg=000000$.

We make use of the following facts : -

There is a unique homomorphism from $latex {\mathbb{Z}\rightarrow R}&fg=000000$ where $latex {R}&fg=000000$ is a ring with unity. If $latex {\phi: R\rightarrow R'}&fg=000000$ is a homomorphism then there is a unique homomorphism $latex {\Phi:R[x_1,\ldots,x_n]\rightarrow R'}&fg=000000$ which preserves action of $latex {\phi}&fg=000000$ on constants and $latex {x_i\mapsto \alpha_i'\in R'}&fg=000000$.

Using the above facts there is a unique homomorphism

$latex \displaystyle \psi : \mathbb{Z}[\{x_{ij}\},\{y_{ij}\},\lambda]\rightarrow R[\lambda]&fg=000000$

which sends

$latex \displaystyle x_{ij}\mapsto a_{ij}&fg=000000$

$latex \displaystyle y_{ij}\mapsto b_{ij}&fg=000000$

$latex \displaystyle \lambda \mapsto \lambda&fg=000000$

Thus, $latex {\psi(P)=det(\lambda I - [a_{ij}][b_{ij}]) -det(\lambda I - [b_{ij}][a_{ij}])}&fg=000000$.

But as $latex {P=0}&fg=000000$ in $latex {\mathbb{Z}[\{x_{ij}\},\{y_{ij}\},\lambda]}&fg=000000$

$latex \displaystyle \Phi_{AB}=\Phi_{BA}&fg=000000$

where $latex {A=[a_{ij}]}&fg=000000$.