A Short Note on Certain Vector Spaces Associated with Finite Groups

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Throughout this article $latex G$ is a finite group.

1. **Free vector space generated by $latex G$:** Given a field $latex \mathbb{F}$ we take the set of all formal linear combinations of form $latex \displaystyle{\sum_{g\in G}}a_g g$ where $latex a_g\in \mathbb{F}$ $latex \forall g\in G$. We define addition on the set as
   $latex \displaystyle{\sum_{g\in G}}a_g g+\displaystyle{\sum_{g\in G}}b_g g=\displaystyle{\sum_{g\in G}}(a_g+b_g) g$
   and scalar multiplication as
   $latex \alpha \times (\displaystyle{\sum_{g\in G}}a_g g)=\displaystyle{\sum_{g\in G}}(\alpha \times a_g) g$
   for all $latex \alpha \in\mathbb{F}$. We take $latex \displaystyle{\sum_{g\in G}}0_{\mathbb{F}} g$ to be the zero in the set. This set now forms a vector space called the free vector space generated by $latex G$ over $latex \mathbb{F}$. Clearly, the dimension of this vector space is equal to the order of the $latex G$.
2. **Permutation Representation :** Let $latex X$ be a set with a finite number of elements. Let $latex G$ act on $latex X$. To this set $latex X$ we can associate a vector space spanned by a basis, where each basis element corresponds to a unique element in $latex X$. Hence we can index the basis elements with the elements of $latex X$, i.e. let the basis be $latex (e_x)_{x\in X}$. We can now define a $latex G$-action on this basis and extend this action by $latex G$-linearity to the entire vector space. The action is
   $latex g.e_x=e_{g.x}$
   We call this vector space a permutation representation of $latex G$.
3. **Class Function :** A function $latex f:G\rightarrow \mathbb{F}$, where $latex \mathbb{F}$ is a field said to be a class function if
   $latex f(hgh^{-1})=f(g)$
   for all $latex g,h in G$.

   The set of all class functions for the given group forms a vector space $latex \mathscr{V}$ over $latex \mathbb{F}$. What is the dimension of this vector space? We observe that every class function is invariant on the conjugacy classes of $latex G$. Hence an easy guess would be that the dimension of this vector space is equal to the number of conjugacy classes of $latex G$. After one makes this guess it is very easy to see why it is so by producing a basis for $latex \mathscr{V}$.

Certain Remarks

- One can notice that we haven't used the group structure in 1. Hence, we can extend this definition to any set with a finite cardinality. Infact we can further extend this definition to any set Y where we define formal linear combinations to be $latex \displaystyle{\sum_{y\in Y}}a_y y$ where atmost finitely many of the $latex a_y$'s can be non-zero.
- In 1. we can generalize the construction to make a module over a ring in a similar way. We can infact make a ring by defining the multiplication in a way similar to the multiplication of polynomials. We call the group so formed as the group ring or the group algebra.
- In 3. if the underlying field is algebraically closed and its characteristic doesn't divide the order of $latex G$(this is not the strongest condition though) then the characters of all irreducible representations of $latex G$(all of which are class functions) form an orthogonal basis for the vector space with respect to the following inner product :
  $latex <f_1,f_2>=\displaystyle{\sum_{g\in G}}f_1(g)f_2(g^{-1})$
  where $latex f_1$ and $latex f_2$ are class functions.