An Equilateral Triangle Inside a Square

by Bishal Deb - Tuesday, September 01, 2015

https://gonitsora.com/an-equilateral-triangle-inside-a-square/

This is an article on an olympiad problem. Here we present various solutions of the problem. We show the beauty of this problem by presenting different proofs to the same problem.

The Problem Statement: -

$latex ABCD$ is a square. $latex E$ is a point inside the square such that $latex \angle{ECB}=\angle{EBC}=15^{\circ}$. Show that $latex \Delta AED$ is equilateral.

A Solution by the use of Trigonometry

Construct the perpendicular bisector of $latex BC$ and $latex AD$ and call it $latex MN$. Since $latex \Delta EBC$ is isosceles $latex MN$ passes through $latex E$. Call the length of the side of the square as $latex a$. Now in right angled triangled $latex \Delta EBN$ use $latex BN=\dfrac{a}{2}$ and the tan of $latex \angle{EBN}=\angle{EBD}=15^{\circ}$ to get the length $latex EN$ in terms of $latex a$. Now $latex ME=MN-EN=a-EN$. Hence use this and $latex AM=\frac{a}{2}$ to get $latex \tan(\angle{EAM})$ and hence $latex \angle{EAD}$ which we find equal to $latex 60^{\circ}$. Similarly we can proceed for $latex \angle{EDA}$ and hence we are done.

First Synthetic Solution by Construction

We use proof by contradiction. The idea is that if what is given in the question is right then $latex AD=AE=AB$ are all radii of the same circle with radius equal to the length of the side of the square and with center $latex A$. Let us assume that the point $latex E$ does not fall on the circle. Then the line BE should meet the circle in some other point as it is already meeting it at the point $latex B$ non-perpendicularly. Let that point be $latex E'$. Our aim is to show that $latex E=E'$.

In $latex \Delta AE'B$, $latex AE'=AB$ (radii of the same circle). Hence it gives us $latex \angle{AE'B}=\angle{ABE'}=90^{\circ}-\angle{E'BC}=75^{\circ}$. Hence it leads to $latex \angle{E'AB}=30^{\circ}\Rightarrow \angle{E'AD}=60^{\circ}$. Similarly we get that $latex \Delta AE'D$ is isosceles giving us $latex \Delta AE'D$ equilateral. USing the same construction as in the first solution and using the fact that $latex \Delta AE'D$ is equilateral we get that $latex E'$ lies on $latex MN$ but $latex E'$ also lies on $latex DE$. Hence, $latex E'= MN\cap DE$. But $latex E$ also lies on $latex MN$ and therefore $latex E= MN\cap DE$. Since two non-parallel straight lines can meet at only one point point hence $latex E'=E$, a contradiction. Hence, $latex \Delta AED$ is equilateral.

A Solution by Subtracting

We construct equilateral $latex \Delta AE'D$ inside the square and show that $latex E=E'$.

A Solution by Using Inequalities

We suppose $latex \angle{AEB}=\angle{DEC}=\epsilon$, $latex AB=BC=CD=DA=a$, $latex AE=ED=b$, $latex BE=EC=c$. Then

$latex b<a\Rightarrow\epsilon>75^{\circ}\Rightarrow\alpha<60^{\circ}\Rightarrow\beta>60^{\circ}\Rightarrow b>a$

Similarly we can show contradiction for the case when $latex b>a$. Hence we have contradiction in both the cases. Hence, $latex b=a$.

Another Synthetic Solution

We erect $latex \Delta CDF\cong\Delta BCE$ on $latex BC$ to the interior. We join $latex EF$. Now it is easy to see that $latex DE=a$.

Yet Another Synthetic Solution

We erect regular $latex Delta BCE'$ on $latex BC$ to the exterior. Then $latex \Delta BEE'\cong\Delta CEE'$ are isosceles, i.e., $latex EE'=a$. Since, $latex \Delta BEE'\cong\Delta ABE$ giving us $latex AE=EE'=a$.

Some remarks

The last 4 solutions are there in the books given in the references. The first was 3 are the ones I was able to come up on my own. Prof B.J. Venkatachala mentions here that this problem is one of his favourite problems

Acknowledgements

I would like to thank Prof. B. R. Sharma of Dibrugarh University for introducing me to this beautiful problem.

References

[1] M R Modak, S A Katre, V V Acharya, An Excursion in Mathematics, 7th edition, Bhaskaracharya Pratishthana, 2007.

[2] H S M Coxeter, Introduction to Geometry, 2nd edition, John Wiley & Sons, 1969.

[3] Arthur Engel, Problem-Solving Strategies, Springer, 1999.

[4] H S M Coxeter and S L Greitzer, Geometry Revisited, New Mathematical Library 19, The Mathematical Association of America, 1967.