Apery's constant

by Kushal Bhuyan - Wednesday, April 20, 2016

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Apery's constant is defined as $latex \displaystyle \zeta{(3)}=1.202056\ldots&fg=000000$, where $latex \displaystyle \zeta(x)&fg=000000$ represents the Euler's zeta function. First lets us define the zeta function. Zeta function is defined as $latex \displaystyle \zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\cdots,&fg=000000$ where $latex s$ is a complex number.

The story of Apery's constant began with the famous Basel problem and Euler's solution of it. At the time people were looking for the exact sum of the following series, $latex \displaystyle 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\cdots,&fg=000000$ which is, as one can see, nothing but $latex \displaystyle \zeta(2)&fg=000000$. Euler, with his stroke of genius, proved that, $latex \displaystyle 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\cdots=\frac{\pi^2}{6}.&fg=000000$ And the pattern continues for all other even numbers, as Euler proved. For instance $latex \displaystyle \zeta(4)=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\cdots=\frac{\pi^4}{90},&fg=000000$ $latex \displaystyle \zeta(6)=1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\cdots=\frac{\pi^6}{945}&fg=000000$ etc. A natural question arises now. What is the value of $latex \displaystyle \zeta(3)=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\frac{1}{6^3}+\cdots &fg=000000$ or the case for any odd number? We might expect that sum of reciprocals of the cubes to be a rational multiple of $latex \displaystyle \pi^3&fg=000000$ or the sum of reciprocals of fifth power to be rational multiple of $latex \displaystyle \pi^5&fg=000000$ and so on. But Euler himself was silent in this case so were many mathematicians since then. Nobody knows if there exists any closed forms for the values of $latex \displaystyle \zeta(k)&fg=000000$ where $latex \displaystyle k&fg=000000$ is an odd number. Euler's proof for $latex \displaystyle \zeta(n)&fg=000000$ for all even $latex \displaystyle n&fg=000000$ was very elegant and innovative but unfortunately those ideas were no good for odd numbers. Curious readers can go through the book *Journey Through Genius* by William Dunham for the proofs of Euler.

Apery's constant is the numerical value of $latex \displaystyle \zeta(3)=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\frac{1}{6^3}+ \cdots=1.202056903159594285399738161511449990764986292\ldots&fg=000000$. As we know that $latex \displaystyle \pi&fg=000000$ is irrational (in fact transcendental), so the value of all $latex \displaystyle \zeta(2n)&fg=000000$ are irrational numbers. So it is not wrong for someone to think that same might hold for the values of $latex \displaystyle \zeta(2n+1)&fg=000000$. Particularly speaking, $latex \displaystyle \zeta(3)&fg=000000$ may be irrational as well. Since all the values of $latex \displaystyle \zeta(2n)&fg=000000$ are rational multiples of powers of $latex \displaystyle \pi&fg=000000$, so in a similar line of thought, if we divide the Apery's constant by $latex \displaystyle \pi^3&fg=000000$, we get, $latex \displaystyle 0.03876817960291679894111989031872114\ldots&fg=000000$, this number does not show any sign of recurring, so it looks irrational. But in mathematics, one can never say conclusively anything by looking at a computational result. We need a proof, a hard and rigorous proof, which settles the question completely.

The man who came to the rescue was **Roger Apery** (14 November 1916 – 18 December 1994). Apery was a Greek-French mathematician, born in Rouen. In 1979, he published a proof of the irrationality of $latex \displaystyle \zeta(3)&fg=000000$, to everybody's surprise. The techniques that Apery used, were well known in Euler's time as well. (Surprisingly Euler missed the proof, otherwise we would have another constant named after him). Many proofs has been found since. Now it is confirmed that $latex \displaystyle \zeta(3)&fg=000000$ is irrational. But it is still not known wether Apery's constant is transcendental or not. Also the methods used in the proof of irrationality of $latex \displaystyle \zeta(3)&fg=000000$ don't seem to extend to other odd integers.

Within a year of Apery's proof, another proof was found by Frits Beukers. A recent proof was found by Wadim Zudilin and a fourth proof is by Yuri Nesterenko. In 2000, Tanguy Rivoal showed that infinitely many of the numbers $latex \displaystyle \zeta(2n + 1)&fg=000000$ must be irrational. In 2001, Wadim Zudilin proved that at least one of the numbers $latex \displaystyle \zeta(5), \zeta(7), \zeta(9)&fg=000000$, and $latex \displaystyle \zeta(11)&fg=000000$ must be irrational. Many people computed the digits of Apery's constant upto billions of digits, using computers. Recently, in November 2015, Dipanjan Nag calculated upto 400,000,000,000 digits.

There are many other representations of Apery's constant, but we conclude this article with the representation found by Ramanujan, $latex \displaystyle \zeta(3)=\frac{7}{180}\pi^3-2\sum_{k=1}^{\infty}\frac{1}{k^3(\exp{2\pi k}-1)}.&fg=000000$

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