Basic of Mathematical Modeling

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Mathematical Model is the expression that we reflect a problem in mathematical language. Every individual literate or illiterate uses it. We all are applying it in our day-to-day life. If we know how to make a model to a situation it makes easy to tackle the situation.

However, there is some way to construct a model. It is important to study it as a discipline. The scientists, engineers are using it successfully throughout the ages. Now it has been realized to study and cultivate it in a proper way. Therefore, mathematics is now expanding its horizon. It is not only for scientists now. It goes to economics, to health science, to biology, to social science and to everywhere.

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A mathematical model usually describes a system by a set of variables and parameters and a set of equations that establish the relationship between the variables. The value of the variables is practically anything; real numbers, boolean values, strings etc for example. The variables represent some properties of the system like timing data, some outputs and so on.

The technique of Mathematical Modeling:

Mathematical Modeling essentially consists of translating real world problem into mathematical problem, solving the mathematical problems and interpreting these solutions in the language of real world. Nevertheless, a real world problem, in all its generality can seldom be translated into a mathematical problem, still it works. For example, a ball is the concept of a sphere. However, at every point on the surface the radii are not equal. Moreover, this happens for every real world problem. There is always an error. But we can not be a step ahead without this concept. To make the ball we must know the surface area of a sphere so that we can have enough canvas for making it.

Sometimes we simplify the real world problem and approximate it to another problem that is quite close to original problem and then it is solved. In this case, care must be taken such that all essential features of the problem would be retained. Say for example when we study motion of planets, in most cases we neglect their sizes and structures. We consider them as point masses.

Mathematical modeling problems are often classified into black box or white box models according to how much information is available of the system. A black box model is a system where there is no information available. A white box model is a system where all necessary information is available. Practically all systems are somewhere between black box and white box models. White box models are usually considered easier; but in real world, we rarely encounter to such problem. In case of black box models we try to use functions as general as possible to cover all different models. An often-used approach for black box model is neural network.
Sometimes it is useful to incorporate subjective information into a mathematical model. This can be done based on intuition, experience or experts’ opinion or based on convenience of mathematical form. This may happen in modeling behavior of a particular machine or some facts in nature.

When constructing a model the following points to be noted:

- Have a clear understanding about the real world problem.
- Have an initial insight into the situation by collecting some data and analyzing it. All possible parameters involving physical, chemical, biological, social or economic relevant to the situation must be counted.
- Formulate the problem.
- The variables and parameters to be named and to be classified as known and unknowns.
- Find out the most appropriate mathematical model and translate the problem to mathematical language.
- Find the solution. It will give a prediction. Compare the prediction with available observation or data. If agreement is good then the model is acceptable. If it is not good change some assumption in the light of the discrepancies observed.

The last point is important. A model must bear the possibility of its improvement in the light of the experimental or observational data. One should be careful such that it is as realistic as possible to represent reality as clearly as possible.

Sometimes we need to develop several models one after another each more realistic than the preceding and likely to be followed by a better one. Sometimes it happens that comparison of prediction with observation reveals the need for new experiments to collect needed data. Therefore, it may develop new concepts. If known mathematical techniques are not adequate to deduce results, it is necessary to develop new techniques; thus adding new concepts to mathematics. Graph Theory, Game Theory, Chaos Theory, Fuzzy Mathematics, Actuarial Science are examples of such grown branches of Mathematics.

**Examples on Mathematical Model:**

1. **Tower of Hanoi:** There are three pegs. On the first one, there is a stack of \( n \) discs of different radii (with holes at the centre) arranged in the descending order starting from the largest disc at the bottom to the smallest disc at the top. The problem is to move these discs from one peg to another at a time and to stack them on another peg in the same order. But the catch is that at no time can a larger disc be placed over a smaller one on the same peg. Obviously if \( n > 1 \), the transfer would be impossible without a third peg.

What is the minimum number of move?

We can design a mathematical model using recurrence relation.

\[
\begin{align*}
\text{ } & \\
\text{a}_{n} & \text{is the minimum number of move for } \text{n discs}.
\end{align*}
\]

\[
\begin{align*}
\text{a}_{0} & = 0 & \text{a}_{1} & = 1
\end{align*}
\]
\[ a_n = 2a_{n-1} + 1 \] for \( n > 1 \)

Also it is \[ a_n = 2^n - 1 \] for all \( n \).

2. Königsberg Bridge Problem: There are four landmasses – two islands and two riverbanks. Seven bridges are there connecting the lands over the river. The problem is to start from any point in one of the landmasses, cover each of the seven bridges once and once only and return to the starting point.

The answer of this problem is no. This question arose among the intellectuals of the city Königsberg. It was the Pregel River flowing through the city with two islands. Many people tried unsuccessfully, until Euler, proved it. A new branch – Graph Theory emerged.

It was proved using graph theory model that the walk is impossible.

3. Model of rational behavior of a consumer: We assume a consumer faces a choice of \( n \) commodities labeled 1, 2, 3, \ldots, \( n \) each with a market price \( p_1, p_2, p_3, \ldots, p_n \). Utility function for the consumer is \( U \) for consumed commodities \( x_1, x_2, x_3, \ldots, x_n \). He has a budget \( M \).

It is an optimization problem. Model is:

\[
\text{max} \quad U(x_1, x_2, x_3, \ldots, x_n)
\]

subject to: \[ \sum_{i=1}^{n} p_i x_i \leq M, \quad \forall i \in \{1, 2, 3, \ldots, n\} \]

4. An Epidemic Model: Let \( S(t) \) and \( I(t) \) be the number of susceptible and infected persons. Initially let there be \( n \) susceptible and one infected person in the system so that

\[
S(t) + I(t) = n + 1, \quad S(0) = n \quad \text{and} \quad I(0) = 1
\]

The function applicable is differential operator. The relation is given by

\[
\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI
\]

5. An Electric Circuit: The current \( i(t) \) ampere represents the time rate of change of charge \( q \) flowing in the circuit. There is a resistance \( R \) ohm in the circuit. There is an inductance \( L \) henry which produces a
potential drop $E_L$. There is a capacitance $C$ which produces a potential drop $E_C$. A battery of $E$ volt is fitted to the circuit.

The expression for voltage is fitted to differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Conclusion: To construct a model is not so easy. As the system goes complicated, more errors are likely to occur. So one should keep in mind all possible factors and then go through it.

**Reference:** Mathematical Modelling by J N Kapur

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