Construction of Real Numbers Using Dedekind Cuts

by Surya Sekhar Bose - Tuesday, June 12, 2018

http://gonitsora.com/construction-of-real-numbers-using-dedekind-cuts/

We all know about rational numbers. Informally, these are ratios of integers, i.e., numbers of the form $latex {a/b}&fg=000000$ where $latex {a}&fg=000000$ and $latex {b}&fg=000000$ are integers (with $latex {b\ne0}&fg=000000$ of course). Any two rational numbers $latex {p, q}&fg=000000$ have a sum $latex {p+q,}&fg=000000$ a product $latex {p\cdot q,}&fg=000000$ and there is an order relation $latex {\le}&fg=000000$ such that $latex {p\le q}&fg=000000$ or $latex {q\le p.}&fg=000000$ These operations $latex {+, \cdot}&fg=000000$ and the relation $latex {\le}&fg=000000$ obey certain properties, which makes the set $latex {\mathbb{Q}}&fg=000000$ of rational numbers an ordered field.

Using this system we can solve most of the elementary problems in mathematics and related disciplines. But there are issues this system can't address. One such problem is the classical one which the Greeks encountered. *Measure the length of the hypotenuse of a right-angled triangle whose other two sides are of unit length.* Taking $latex {x}&fg=000000$ as the required length, the Pythagorean theorem gives us $latex {x^2=2.}&fg=000000$ Now a simple and well-known argument shows that $latex {x\not\in\mathbb{Q}.}&fg=000000$ Let us place the hypotenuse on the $latex {x}&fg=000000$-axis in such a way that one end coincides with the origin $latex {O}&fg=000000$ and the other end $latex {A}&fg=000000$ lies on the positive $latex {x}&fg=000000$-axis. Then $latex {A}&fg=000000$ does not correspond to a rational number; it corresponds to something else.

So we see that the rational numbers are not sufficient for the purpose of measuring lengths of straight line segments; we need new numbers for this. As you have probably already guessed, these new numbers are the irrational numbers. Together with the rational numbers they form the set $latex {\mathbb{R}}&fg=000000$ of real numbers. The operations $latex {+, \cdot}&fg=000000$ and the relation $latex {\le}&fg=000000$ on $latex {\mathbb{Q}}&fg=000000$ can be extended to $latex {\mathbb{R},}&fg=000000$ and the resulting system is said to be the *system of real numbers*. We shall construct this system in two different ways: by Dedekind cuts, and by Cauchy sequences (to be disussed in a subsequent post).

We shall now construct the set of real numbers using what are called **Dedekind Cuts**. This construction is named after the German mathematician Richard Dedekind. We see that the point $latex {A}&fg=000000$ does not correspond to a rational number. This means each rational number either lies on the left of $latex {A}&fg=000000$ or on the right of $latex {A.~A}&fg=000000$ ``cuts'' the set $latex {\mathbb{Q}}&fg=000000$ of rational numbers into two halves: $latex {C,}&fg=000000$ the set of those rational numbers that lie on the left of $latex {A,}&fg=000000$ and $latex {D,}&fg=000000$ the set of those rational numbers that lie on the right of $latex {A}&fg=000000$. The partition $latex {(C, D)}&fg=000000$ of $latex {\mathbb{Q}}&fg=000000$ is said to be a *Dedekind cut*.

The set $latex {C}&fg=000000$ satisfies the following properties.

1. (1) $latex {C}&fg=000000$ is a nonempty proper subset of $latex {\mathbb{Q}.}&fg=000000$

2. (2) If $latex {p\in C}&fg=000000$ and $latex {q\in\mathbb{Q}}&fg=000000$ such that $latex {q<p,}&fg=000000$ then $latex {q\in C.}&fg=000000$
3. (3) For every $latex {p\in C}&fg=000000$ there exists $latex {r\in C,}&fg=000000$ such that $latex {p<r.}&fg=000000$

The second property says that every rational number less than some element of $latex {C}&fg=000000$ is also an element of $latex {C.}&fg=000000$ The third property says that $latex {C}&fg=000000$ has no largest element. We let $latex {\mathbb{R}_D}&fg=000000$ denote the set of those subsets of $latex {\mathbb{Q}}&fg=000000$ that satisfy properties (1), (2) and (3). Observe that each element $latex {C\in\mathbb{R}_D}&fg=000000$ corresponds to the Dedekind cut $latex {(C, \{p\in\mathbb{Q}: q<p~\mbox{for~every~}q\in C\})}&fg=000000$ and conversely. Since we can intuitively see that each Dedekind cut corresponds to a unique point on a straight line and conversely, the elements in $latex {\mathbb{R}_D}&fg=000000$ corresponds to the points on a straight line. Observe that every element $latex {d\in D}&fg=000000$ satisfies $latex {c\le d}&fg=000000$ for all $latex {c\in C;}&fg=000000$ in other words, every element of $latex {D}&fg=000000$ is an upper bound of $latex {C.}&fg=000000$ But there is no least upper bound because $latex {D}&fg=000000$ has no smallest element.

We shall extend the operations $latex {+, \cdot}&fg=000000$ and the relation $latex {\le}&fg=000000$ from $latex {\mathbb{Q}}&fg=000000$ to $latex {\mathbb{R}_D}&fg=000000$ to create *a* system of real numbers. Note that these extensions will really be extensions only if $latex {\mathbb{Q}\subset\mathbb{R}_D.}&fg=000000$ Strictly speaking, we don't have $latex {\mathbb{Q}\subset\mathbb{R}_D;}&fg=000000$ because the element in $latex {\mathbb{R}_D}&fg=000000$ are subsets of $latex {\mathbb{Q},}&fg=000000$ whereas the elements of $latex {\mathbb{Q}}&fg=000000$ are not. To get around this we identify $latex {p\in\mathbb{Q}}&fg=000000$ with the set $latex {\{r\in\mathbb{Q}:r<p\}.}&fg=000000$ With this identification we have $latex {\mathbb{Q}\subset\mathbb{R}_D.}&fg=000000$ Now first we extend the order relation $latex {\le}&fg=000000$ from $latex {\mathbb{Q}}&fg=000000$ to $latex {\mathbb{R}_D.}&fg=000000$ For $latex {r, s\in\mathbb{R}_D}&fg=000000$ we define $latex {r\le_Ds}&fg=000000$ iff $latex {r\subseteq s}&fg=000000$ (recall that $latex {r, s}&fg=000000$ are the subsets of $latex {\mathbb{Q}}&fg=000000$). Using the properties (1), (2) and (3) one can prove that the relation $latex {\le_D}&fg=000000$ is reflexive, anti-symmetric and transitive, and that $latex {r\le_Ds}&fg=000000$ or $latex {s\le_Dr.}&fg=000000$ Thus $latex {\le_D}&fg=000000$ is a legitimate order relation on $latex {\mathbb{R}_D.}&fg=000000$ But why is it an extension of the relation $latex {\le}&fg=000000$ on $latex {\mathbb{Q}?}&fg=000000$ If $latex {r, s\in\mathbb{Q},}&fg=000000$ then (because of our identification) $latex {r=\{p\in\mathbb{Q}:p<r\},~s=\{p\in\mathbb{Q}:p<s\}.}&fg=000000$ A moment of reflection convinces one that $latex {r\subseteq s}&fg=000000$ iff $latex {r\le s.}&fg=000000$ Hence $latex {\le_D}&fg=000000$ is an extension of $latex {\le.}&fg=000000$ There is one property that $latex {\le_D}&fg=000000$ has which $latex {\le}&fg=000000$ does not have: The least upper bound property. It says that every non-empty subset of $latex {\mathbb{R}_D}&fg=000000$ that has some upper bound has a smallest upper bound. This removes the difficulty we had with $latex {C}&fg=000000$ and $latex {D}&fg=000000$ in the previous paragraph. It can be easily proved. Consider a non-empty subset $latex {S\subseteq\mathbb{R}_D.}&fg=000000$ Let $latex {S}&fg=000000$ have an upper bound $latex {t\in\mathbb{R}_D.}&fg=000000$ This means $latex {s\subseteq t}&fg=000000$ for every $latex {s\in S,}&fg=000000$ and hence $latex {s\subseteq\displaystyle\bigcup_{s\in S}s\subseteq t}&fg=000000$ for every $latex {s\in

S.}&fg=000000$ To complete the proof, i.e., to show that $latex {\displaystyle\bigcup_{s\in S}s}&fg=000000$ is the least upper bound of $latex {S,}&fg=000000$ one has to prove that $latex {\displaystyle\bigcup_{s\in S}s\in\mathbb{R}_D.}&fg=000000$ That is easy. This relatively easy proof is an advantage of this construction over the other construction involving Cauchy sequences.

Extension of the operations $latex {+, \cdot}&fg=000000$ from $latex {\mathbb{Q}}&fg=000000$ to $latex {\mathbb{R}_D}&fg=000000$ are less trivial and are done as follows. For $latex {r, s\in\mathbb{R}_D}&fg=000000$ we define

- $latex r+_Ds :=\{p+q:p\in r, q\in s\}$.
- $latex -r :=\{p-q:p<0, q\in\mathbb{Q}\setminus r\}$.
- If $latex r, s\geq 0$ then $latex r\cdot_Ds :=\{pq:p\geq 0, p\in r, q\geq 0, q\in s\}\cup\{p\in\mathbb{Q}:p<0\}$; and$latex r\cdot_D s= -[(-r))\cdot_D s]$, if $latex r<0, s\geq 0$, or $latex r\cdot_D s = -[r\cdot_D (-s)]$, if $latex r\geq 0, s< 0$, or $latex r\cdot_D s= -(-r)\cdot_D (-s)$, if $latex r<0, s< 0$.
- If $latex r>0,$ then $latex 1/r := \{1/p:p\in\mathbb{Q}\setminus r\}$; if $latex r<0$, then $latex 1/r :=-(1/(-r))$.

It can be proved that $latex {(\mathbb{R}_D, +_D, \cdot_D, \le_D)}&fg=000000$ is an ordered field; because of the least upper bound property $latex {(\mathbb{R}_D, +_D, \cdot_D, \le_D)}&fg=000000$ is a complete ordered field. Further, it has $latex {(\mathbb{Q}, +, \cdot, \le)}&fg=000000$ as a subsystem.
