Exceptional Embedding of S5 in S6

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In this article we shall show a peculiar property of $latex S_5$.

We first note that there are several subgroups of $latex S_n$ isomorphic to $latex S_{n-1}$. In fact there is one for each $latex 1\leq i\leq n$
$latex H_i=\{ \sigma \in S_n |\sigma (i) = i \}$

Let $latex X=\{1,2,3,4,5,6\}$.

All of the above subgroups have two orbits in $latex X$. But for $latex S_6$ we show that there is a seventh embedding of $latex S_5$ which acts transitively on $latex X$.

We now show that there is a transitive action of $latex S_5$ on X.

We look at the 5-Sylow subgroups of $latex S_5$. They will be of order 5 each as $latex S_5$ is of order 120 and 5 is the highest power of 5 dividing it. Hence, the only possibility for the 5-Sylow subgroups is to contain 5 cycles. Now, the number of r cycles of $latex S_n$ is $latex \binom{n}{r}(r-1)!$. Hence there are 24 5-cycles. As each 5-Sylow subgroup consists of 4 cycles and as any two 5-Sylow subgroups has only identity in their intersection so number of 5-Sylow subgroups is $latex 24/4=6$. Let the subgroups be $latex P_1, \ldots,P_6$. The number of subgroups is compatible with the result of Sylow's theorem which says that the number of 5-Sylow subgroups should be of the form $latex 5k+1$ and should divide 120.

Let $latex Syl_5(S_5):=\{P_1,\ldots, P_6\}$. We define an action of $latex S_5$ on this set by conjugation. By Sylow's theorem this is an action. Infact it is a transitive action. As $latex Syl_5(S_5)$ and $latex X$ are bijective, we have established an action of $latex S_5$ on $latex X$.

Let $latex \phi : S_5\rightarrow Perm(X)$ be the action homomorphism. If we show that $latex \phi$ is injective then we see that there is an embedding of $latex S_5$ in $latex S_6$ which acts transitively on $latex X$.

Now we show that $latex \phi$ is injective.

Showing $latex \phi$ is injective is equivalent to showing that $latex Ker(\phi)$ is $latex \{1\}$. Now $latex Ker(\phi)$ is a normal subgroup of $latex S_5$. Hence $latex Ker(\phi)$ is either $latex \{1\}$, $latex A_5$ or $latex S_5$. Here we use the fact that $latex A_n$ is simple for all $latex n\geq 5$. We know that $latex S_5/Stab(P_1)$ is isomorphic to $latex orb(P_1)$. As $latex Ker(\phi)\subset Stab(P_1)$ so if $latex Ker(\phi)$ is either $latex A_5$ or $latex S_5$ then $latex |orb(P_1)|\leq 2$ but $latex |orb(P_1)|=6$ as the action is transitive. Hence $latex Ker(phi)$ is $latex \{1\}$.

It can be further showed that only n for which $latex S_n$ has a subgroup isomorphic to $latex S_{n-1}$ which acts transitively on $latex \{1,\ldots,n\}$ is 6.

The above article is based on a lecture given at Chennai Mathematical Institute by Prof P. Vanchinathan.

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