Figurate Numbers

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https://gonitsora.com/figurate-numbers/

Elena Deza and Michel Marie Deza

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This book is about special types of numbers (integers) that have geometric associations and that have intriguing spatial properties. The ancient Greeks were perhaps the first to study what are called "figurate numbers" — numbers that can be represented by regular geometric patterns of points in the plane or in space, such as triangular, polygonal and polyhedral numbers. The first two chapters contain a lot of formulae for all kinds of figurate numbers that arise from geometric patterns in 2 and 3 dimensions. Properties and relations between such figurate numbers and their connections with Diophantine equations have been studied by classical mathematicians like Euler, Fermat, Lagrange, Legendre, Cauchy, Gauss and Dirichlet.

Chapter 3 extends the construction of figurate numbers to dimension 4 and beyond. Examples of such numbers are the pentatope numbers which are 4-dimensional analogues of triangular and tetrahedral numbers, and the biquadratic numbers which are the 4-dimensional analogues of square and cubic numbers. Despite the lack of visual pictures and physical intuition, multitudes of formulae are presented and proved.

Chapter 4 contains much interesting material on the role of certain figurate numbers in classical number theory. One finds connections with well-known numbers associated with the names of Catalan, Mersenne, Fermat, Fibonacci, Lucas, Stirling, Bernoulli, Bell and so on. Certain types of Diophantine equations inevitably turn up — the Fermat equation, Pell equation, Ramanujan–Nagell equation. We get to see some recreational aspects of prime numbers in terms of square arrangements of their digits. Most people have come across magic squares and magic cubes, but probably not magic hexagons. There is a brief mention of unrestricted partitions and Waring's problem.

The first four chapters may appear to be a collection of results and properties about "exotic" numbers and lack a general theory. However, from a number-theoretic point of view, Chapter 5 is the most interesting and modern. It revolves around Fermat's polygonal number theorem which was mentioned in a letter from Fermat to Mersenne as the statement that every integer is a sum of at most 3 triangular numbers, 4 squares, 5 pentagonal numbers and so on. In his usual style, Fermat hints at a proof for which he would need to devote an entire book, but such a book or proof has yet to be discovered. Unlike Fermat's more famous "Last Theorem", his polygonal number theorem was proved by Cauchy around 1813 and a simplified proof was given by T Pepin in 1892. Apparently the book under review is the first book to present the full details of the proof. Before that, results about the polygonal number theorem are scattered among hard to access publications and defunct journals. In the classical connection, we also come across the names of illustrious mathematicians like Gauss, Lagrange, Legendre, Jacobi, Dirichlet and Minkowski. The bridge between the classical and the modern setting was perhaps provided by L E Dickson, and the final *coup de grace* was delivered by M B Natanson in 1987.

There is a list of figurate-related numbers in Chapter 6 — a kind of zoo of numbers with "exotic" and unusual numerical properties, like rare trophies of number hunters and collectors; for example, the largest known prime that cannot be expressed as a sum of 3 hexagonal numbers, or a prime that can be represented as a sum of a triangular number and a perfect square in 27 distinct ways. The list is probably not exhaustive; there must be numerous puzzles in recreational mathematics that would be a good test bed for arithmetic ingenuity and computer skills. In fact, the last chapter contains 155 problems (with solutions given) to indulge in one's fascination for curious numerical properties which could have also occupied mathematicians as a pastime.

This book offers a potential source of information for those interested in numbers and numerical properties associated with geometric configurations. It collects together a wide range of results scattered throughout the literature and gives numerous references to books on recreational mathematics as well as elementary number theory. However, there are many typographical errors in language — the proof reading could have been more thorough. In many places also, the language is awkward and sometimes incorrect (probably due to the fact that the authors are not English native-speaking) though this may not be mathematically significant. All in all, it is good to know that there is a book one can turn to if you need to check up some lesser known things about numbers associated with geometric patterns.

Y K Leong

National University of Singapore.

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