Fundamental Theorem of Algebra using Galois theory

by Bishal Deb - Wednesday, November 30, 2016

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In this article we shall prove that $latex \mathbb{C}$ is algebraically closed. Here we consider $latex \mathbb{C}$ as a splitting field of the polynomial $latex x^2+1$.

The proof uses very little analysis and most of it is Galois theory. The only facts from analysis which will be used in proof are :

- Positive real numbers have square roots.
- Every polynomial of odd degree with real coefficients has a real root.

Both of these facts are consequences of the intermediate value theorem.

We first show that every element of $latex \mathbb{C}$ has a square root in $latex \mathbb{C}$. For $latex a,b\in\mathbb{R}$ let

$latex c=\pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$

$latex d=\pm\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$

We choose the sign of $latex c$ and $latex d$ such that $latex cd$ has the same sign as b. Then we can see that $latex (c+\iota d)^2=a+\iota b$.

Now we show that every polynomial $latex f(x)\in \mathbb{R}[x]$ splits in $latex \mathbb{C}$. This is equivalent to showing that the splitting field of $latex f(x)(x^2+1)$ over $latex \mathbb{R}$ is $latex \mathbb{C}$. Let $latex E$ be the splitting field of $latex f(x)(x^2+1)$ over $latex \mathbb{R}$ for a fixed $latex f$. As $latex \mathbb{R}$ has characteristic zero, $latex f(x)(x^2+1)$ is a separable polynomial and hence $latex E/\mathbb{R}$ is a Galois extension. Let $latex G=Gal(E/\mathbb{R})$.

Let $latex H$ be a 2-Sylow subgroup of $latex G$. Let $latex M=E^H$, the subfield of E fixed by $latex H$. Then degree of the extension $latex M/\mathbb{R}$ is $latex |G/H|$ which is odd. Hence, for any $latex \alpha\in M$ it's minimal polynomial over $latex \mathbb{R}$ has odd degree. But as any odd degree polynomial has a root in $latex \mathbb{R}$, $latex \alpha\in\mathbb{R}$. Thus, $latex M=\mathbb{R}$ and hence $latex G$ is a 2-group.

If $latex G\neq (1)$ it has a subgroup $latex N$ of index $latex 2$. Then degree of $latex E^N$ over $latex \mathbb{C}$ is 2 and hence is generated by the square root of an element in $latex \mathbb{C}$. But as $latex \mathbb{C}$ has all its square roots we get that $latex G=(1)$. Hence, $latex E=\mathbb{R}$

Source : Field and Galois Theory Notes, James Milne
