Hilbert's Axioms of Geometry

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David Hilbert was a German mathematician who is known for his problem set that he proposed in one of the first ICMs, that have kept mathematicians busy for the last century. Hilbert is also known for his axiomatization of the Euclidean geometry with his set of 20 axioms. These axioms try to do away the inadequacies of the five axioms that were postulated by Euclid around two millenia ago. In particular Hilbert does away with most of the problems of the fifth postulate of Euclid which many before him and since have thought to be incomplete. Apart from Hilbert there were other mathematicians like Veblen, Tarski, Birkoff, Moore etc who also gave alternate sets of axioms. These are left to be discussed in a later post. Below we gave verbatim the list of Hilbert's Axioms (sourced from Wikipedia):

I. Combination

1. Two distinct points *A* and *B* always completely determine a straight line *a*. We write *AB* = *a* or *BA* = *a*. Instead of “determine,” we may also employ other forms of expression; for example, we may say “*A* lies upon *a*”, “*A* is a point of *a*”, “*a* goes through *A* and through *B*”, “*a* joins *A* to *B*”, etc. If *A* lies upon *a* and at the same time upon another straight line *b*, we make use also of the expression: “The straight lines *a* and *b* have the point *A* in common,” etc.
2. Any two distinct points of a straight line completely determine that line; that is, if *AB* = *a* and *AC* = *a*, where *B* ? *C*, then also *BC* = a.
3. Three points *A*, *B*, *C* not situated in the same straight line always completely determine a plane ?. We write *ABC* = ?. We employ also the expressions: “*A*, *B*, *C*, lie in ?”; “A, B, C are points of ?”, etc.
4. Any three points *A*, *B*, *C* of a plane ?, which do not lie in the same straight line, completely determine that plane.
5. If two points *A*, *B* of a straight line *a* lie in a plane ?, then every point of *a* lies in ?. In this case we say: “The straight line *a* lies in the plane ?,” etc.
6. If two planes ?, ? have a point *A* in common, then they have at least a second point *B* in common.
7. Upon every straight line there exist at least two points, in every plane at least three points not lying in the same straight line, and in space there exist at least four points not lying in a plane.

II. Order

1. If a point *B* is between points *A* and *C*, *B* is also between *C* and *A*, and there exists a line containing the points *A*,*B*,*C*.
2. If *A* and *C* are two points of a straight line, then there exists at least one point *B* lying between *A* and *C* and at least one point *D* so situated that *C* lies between *A* and *D*.
3. Of any three points situated on a straight line, there is always one and only one which lies between the other two.

4. Pasch's Axiom: Let A, B, C be three points not lying in the same straight line and let *a* be a

straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line *a* passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

III. Parallels

1. In a plane ? there can be drawn through any point *A*, lying outside of a straight line *a*, one and only one straight line which does not intersect the line *a*. This straight line is called the parallel to *a* through the given point *A*.

IV. Congruence

1. If *A*, *B* are two points on a straight line *a*, and if *A?* is a point upon the same or another straight line *a?* , then, upon a given side of *A?* on the straight line *a?* , we can always find one and only one point *B?* so that the segment *AB* (or *BA*) is congruent to the segment *A?B?* . We indicate this relation by writing *AB* ? *A? B?*. Every segment is congruent to itself; that is, we always have *AB* ? *AB*.
   We can state the above axiom briefly by saying that every segment can be *laid off* upon a given side of a given point of a given straight line in one and only one way.
2. If a segment *AB* is congruent to the segment *A?B?* and also to the segment *A?B?*, then the segment *A?B?* is congruent to the segment *A?B?*; that is, if *AB* ? *A?B?* and *AB* ? *A?B?*, then *A?B?* ? *A?B?*
3. Let *AB* and *BC* be two segments of a straight line a which have no points in common aside from the point *B*, and, furthermore, let *A?B?* and *B?C?* be two segments of the same or of another straight line *a?* having, likewise, no point other than *B?* in common. Then, if *AB* ? *A?B?* and *BC* ? *B?C?*, we have *AC* ? *A?C?*.
4. Let an angle (h, k) be given in the plane ? and let a straight line a? be given in a plane ??. Suppose also that, in the plane ??, a definite side of the straight line *a?* be assigned. Denote by *h?* a half-ray of the straight line *a?* emanating from a point *O?* of this line. Then in the plane ?? there is one and only one half-ray *k?* such that the angle (*h*, *k*), or (*k*, *h*), is congruent to the angle (*h?*, *k?*) and at the same time all interior points of the angle (*h?*, *k?*) lie upon the given side of *a?*. We express this relation by means of the notation ?(*h*, *k*) ? (*h?*, *k?*)
   Every angle is congruent to itself; that is, ?(*h*, *k*) ? (*h*, *k*)
   or
   ?(*h*, *k*) ? (*k*, *h*)
5. If the angle (*h*, *k*) is congruent to the angle (*h?*, *k?*) and to the angle (*h?*, *k?*), then the angle (*h?*, *k?*) is congruent to the angle (*h?*, *k?*); that is to say, if ?(*h*, *k*) ? (*h?*, *k?*) and ?(*h*, *k*) ? (*h?*, *k?*), then ?(*h?*, *k?*) ? (*h?*, *k?*).
6. If, in the two triangles ABC and A?B?C? the congruences AB ? A?B?, AC ? A?C?, ?BAC ? ?B?A?C? hold, then the congruences ?ABC ? ?A?B?C? and ?ACB ? ?A?C?B? also hold.

V. Continuity

1. Axiom of Archimedes. Let A_1 be any point upon a straight line between the arbitrarily chosen points *A* and *B*. Take the points A_2, A_3, A_4, . . . so that A_1 lies between *A* and A_2, A_2 between A_1 and

A_3, A_3 between A_2 and A_4 etc. Moreover, let the segments AA_1, A_1A_2, A_2A_3, A_3A_4, . . . be equal to one another. Then, among this series of points, there always exists a certain point A_n such that B lies between A and A_n.

2. *Axiom of completeness*. To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.
