Infinity Comparision

by Gonit Sora - Wednesday, October 01, 2014

https://gonitsora.com/infinity-comparision/

What is infinity? Is it a number or just another mathematical definition to fit our purpose? But then what are numbers? It is pointless asking such questions. The answers are some philosophical thoughts. Then how do we deal with them whenever they arise?

Well one question which occured in the mind of Cantor is that are all the infinities equal? Or is some kind of infinity bigger than the other kind? Firstly infinity is something which is beyond counting and now we are defining order on something which is bigger than everything else?

Well it comes out that our normal intuition fails when we try to deal with infinities. Lets ask a very simple question. What is more, the number of integers or the number of natural numbers? The obvious answer which comes to mind is that the set of integers seems to be a larger set as it contains the set of natural numbers. But let us examine the situation more carefully :

We shall proceed with the traditional way of trying to form a bijection between the two sets and then as the sets form a bijection therefore they contain the same number of elements. We form a map from integers to natural numbers by sending each positive integer $latex n$ to $latex 2n$ and each non-positive integer $latex n$ to $latex -2n+1$. This clearly forms a bijection.

Hence, although in finite cases if a set A is a proper subset of another set B then $latex |A|<|B|$. But for inifinite cases this may not be true. All the sets which form a bijection with the set of natural numbers are known as countable sets. Hence clearly the set of integers and any infinite subset of integers are countable. We shall see that the set of rationals is also countable.

It would be sufficient if we show that the set of all positive rationals form a bijection with the set of natural numbers. To do this we make a table whose rows and columns are indexed as the natural numbers. In the cell in the $latex i^{th}$ row and $latex j^{th}$ column we insert the number $latex \frac{i}{j}$. Now we begin with the element in the $latex 1^{st}$ column and $latex 1^{st}$ row (which is 1) and map it with 1. Next we go to the element in the $latex 2^{nd}$ row, $latex 1^{st}$ column (which is 2) and map it with $latex 2$. And then we take the element in the $latex 1^{st}$ row and $latex 2^{nd}$ column and map it with $latex 3$. We continue in this manner by selecting consecutive finite diagonals and then we traverse the diagonal and whenever we find a number we map it with the next natural number. But what if a number occurs more than once? For example we get $latex \frac{1}{1}$ and $latex \frac{2}{2}$. So what we do is whenever we encounter a rational number which has occured before we ignore it and go to the next number.

Thus we have formed a bijection with the set of positive rationals and the set of positive integers. Similarly we map the negative rationals to the set of negative integers and zero to zero. Thus we have received a bijection between the set of rationals and the set of integers. But we know that the the integers are bijective with natural numbers and hence so are rationals.

Thus the number of rationals is equal to the number of natural numbers and hence $latex \mathbb{Q}$ is countable.

But is there a set whose number of elements is larger than $latex \mathbb{N}$? Well we shall see that this is true for $latex \mathbb{R}$.

Let us assume that $latex \mathbb{R}$ is countable. Hence there is a sequence that has all the members of $latex \mathbb{R}$. Now we form a real number $latex x$ from this sequence. We look at the $latex n^{th}$ number of the sequence see its $latex n^{th}$ digit after the decimal point(in decimal representation). We make $latex x$ by choosing a different digit in its $latex n^{th}$ position after the decimal point. For example if our sequence is:

$latex x_1 = 3.14159265...$
$latex x_2 = 1.87547798...$
$latex x_3 = 0.00000000...$
$latex .$
$latex .$
$latex .$

We may form $latex x=0.291...$. Notice that the $latex n^{th}$ digit after the decimal point is any digit other than that present in the $latex n^{th}$ digit after the decimal point in $latex x_{n}$. As we considered $latex \mathbb{R}$ as countable so $latex x=x_m$ for some $latex \mathbb{N}$, but the $latex m^{th}$ digit after the decimal in $latex x$ and $latex x_m$ doesn't match. Hence $latex \mathbb{R}$ is not countable.

We remark that as the cardinality of $latex \mathbb{R}$ is larger than that of $latex \mathbb{Q}$, there are more irrational numbers than there are rational numbers.

We call the notion of number of elements as cardinality. The cardinality of a set is $latex A$ is denoted by $latex |A|$ and is the number of elements in $latex A$ if it is finite. If it is countable then is denoted as $latex \aleph_{0}$ (the Hebrew letter "aleph" followed by a $latex 0$ in the subscript). It is to be mentioned here that $latex |\mathbb{R}|=2^{\aleph_{0}}$. $latex \aleph_{1}$ is defined to be the next largest infinity after $latex \aleph_{0}$. A question which has been there since the time of Cantor is that whether $latex 2^{\aleph_{0}}=\aleph_1$ which is equivalent to saying that is there an infinity that lies between the number of naturals and the number of reals? This is the continuum hypothesis and is the first problem of Hilbert's famous 23 problems.

In 1963 Paul Cohen proved that by using the current set theory(Zermelo - Fraenkel set theory) we can neither prove nor disprove the continuum hypothesis. Cohen received Fields medal in 1966.

Another question comes to mind is that is there any infinity which is larger than $latex |\mathbb{R}|$. We see that there are infinitely many infinities.

Well it can be shown that the cardinality of the power set of any set is greater than its power set.

Hence given any infinity we can always find a greater infinity. This is just an existence proof. But the realisation is very difficult. $latex \aleph_0$ is the cardinality of natural numbers, integers, rationals and all other countable sets. $latex 2^{\aleph_0}$ is the cardinality of $latex \mathbb{R}$, $latex \mathbb{C}$, $latex \mathbb{R}^n$, set of points on a curve, planes, spaces and their higher dimensional analogues. $latex 2^{2^{\aleph_{0}}}$ is the cardinality of the set of all curves, set of all planes, set of all spaces and their higher dimensional analogues. But a surprising fact is that although we know that there exist higher inifinities we haven't realized them. I shall finish by quoting George Gamow :

> It seems that the three first infinite numbers are enough to count anything we can think of, and we find ourselves in a position exactly opposite to that of our old friend the Hottentot(a war tribe also known as the Khoikhoi who had no words for numbers beyond three) who had many sons but could not count beyond three!
