Morley's Theorem

by Bishal Deb - Friday, December 09, 2011

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INTRODUCTION:

One of the most elementary theorems in elementary geometry was discovered about 1899 by F. Morley (whose son Cristopher wrote novels such as *Thunder on the Left*). He mentioned it to his friends, who spread it over the world in the form of mathematical gossip. At last, after ten years, a trigonometrical proof by M. Satyanarayan and an elementary proof by M. T. Naraniengar were published.

MORLEY'S THEOREM:

The three points of intersection of the adjacent triangles of the angles of any triangle form an equilateral triangle.

In other words, any triangle ABC yields an equilateral triangle PQR if the angles A, B, C are trisected by AQ and AR, BR and BP, CP and CQ, as figure 1. (Much trouble is experienced if try a direct approach but the difficulties disappear if we work backwards, beginning with an equilateral triangle and building up a general triangle which is afterwards identified with the given triangle ABC.)

On the respective sides QR, RP, PQ of a given equilateral triangle PQR erect isosceles triangles P'QR, Q'RP, R'PQ whose base angles $$alpha, beta, gamma$$ satisfy the equation and the inequalities

$latex \alpha+\beta+\gamma = 120^{\circ}, \alpha \leq 60^{\circ}, \beta \leq 60^{\circ}, \gamma \leq 60^{\circ} .$

Extend the sides of the isosceles triangles below their bases until they meet again in points A, B, C. Since $latex \alpha+\beta+\gamma+60^{\circ}=180^{\circ}, $ we can immediately infer the measurements of some angles, as marked in figure 2. For instance, the triangle ABC is to describe it as lying on the bisector of the angle A at such a distance that

?BIC = 90? + ½A.

Applying this principle to the point P in the triangle P'BC, we observe that the line PP' (which is a median of both the equilateral triangle PQR and the isosceles triangle P'QR) bisects the angle at P'. Also the half angle at P' is 90? - $latex \alpha$ and

?BPC = 180? – $latex \alpha$ = 90?+ (90? - $latex \alpha$).

Hence P is the incenter of the triangle P'BC. Likewise Q is the incenter of Q'CA, and R of R'AB.

Therefore all the three small angles at C are equal; likewise at A and at B. In other words, the angles of the triangle ABC are trisected.

The three small angles at A are each 1/3A = 60? - $latex \alpha$; similarly at B and at C. Thus

$latex \alpha$ = 60? – 1/3A, $latex \beta$ = 60? – 1/3B, $latex \gamma$ = 60? – 1/3C.

By choosing these values for the base angles of our isosceles triangles, we can ensure that the above procedures yield a triangle ABC that is similar to the given triangle.

This completes the proof.

(SOURCE: INTRODUCTION TO GEOMETRY, H.S.M. Coxeter, John Wiley & Sons, INC.)

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