Napier's Constant "e" is Transcendental

by Manjil Saikia - Saturday, May 14, 2011

http://gonitsora.com/napiers-constant-is-transcendental/

This theorem was first proved by Hermite in 1873. The below proof is near the one given by Hurwitz. We at first derive a couple of auxiliary results.

Let $$f(x)$$ be any polynomial of degree $$ mu$$ and $$ F(x)$$ the sum of its derivatives,

$$displaystyle F(x) := f(x)+f'(x)+f''(x)+ldots+f^{(mu)}(x).$$ (1)

consider the product $$ Phi(x) := e^{-x}F(x)$$. The derivative of this is simply

$$displaystyle Phi'(x) equiv e^{-x}(F'(x)-F(x)) equiv -e^{-x}f(x).$$

Applying the mean value theorem to the function $$Phi$$ on the interval with end points 0 and $$ x$$ gives

$$displaystyle Phi(x)-Phi(0) = e^{-x}F(x)-F(0) = Phi'(xi)x = -e^{-xi}f(xi)x,$$

which implies that $$ F(0) = e^{-x}F(x)+e^{-xi}f(xi)x$$. Thus we obtain the

Lemma 1. $$ F(0)e^x = F(x)+xe^{x-xi}f(xi)$$ ($$ xi$$ is between 0 and $$x$$)

When the polynomial $$ f(x)$$ is expanded by the powers of $$ x!-!a$$, one gets

$$displaystyle f(x) equiv f(a)+f'(a)(x!-!a)+f''(a)frac{(x!-!a)^2}{2!}+ldots+
f^{(mu)}(a)frac{(x!-!a)^{mu}}{mu!};$$

comparing this with (1) one gets the

Lemma 2. The value $$ F(a)$$ is obtained so that the polynomial $$ f(x)$$ is expanded by the powers of $$ x!-!a$$ and in this expansion the powers $$x!-!a$$, $$ (x!-!a)^2$$, ..., $$ (x!-!a)^{mu}$$ are replaced respectively by the numbers 1!, 2!,...,$$mu !$$.

Now we begin the proof of the theorem. We have to show that there cannot be any equation

$$displaystyle c_0+c_1e+c_2e^2+ldots+c_ne^n = 0$$ (2)

with integer coefficients $$ c_i$$ and at least one of them distinct from zero. The proof is indirect. Let's assume the contrary. We can presume that $$ c_0 neq 0$$.

For any positive integer $$nu$$, lemma 1 gives

$$displaystyle F(0)e^{nu} = F(nu)+nu e^{nu-xi_{nu}}f(xi_{nu}) quad(0 < xi_{nu} < nu)$$. (3)

By virtue of this, one may write (2), multiplied by $$F(0)$$, as

$$displaystyle c_0F(0)!+!c_1F(1)!+!c_2F(2)!+!ldots!+!c_nF(n) =
-[c_1e^{1-xi_1}f(xi_1)!+!2c_2e^{2-xi_2}f(xi_2)!+ldots+nc_ne^{n-xi_n}f(xi_n)].$$ (4)

We shall show that the polynomial $$ f(x)$$ can be chosen such that the left side of (4) is a non-zero integer whereas the right side has absolute value less than 1.

We choose

$$displaystyle f(x) := frac{x^{p-1}}{(p-1)!}[(x!-!1)(x!-!2)cdots(x!-!n)]^p,$$ (5)

where $$ p$$ is a positive prime number on which we later shall set certain conditions. We must determine the corresponding values $$ F(0)$$, $$F(1)$$, ..., $$F(n)$$.

For determining $$ F(0)$$ we need, according to lemma 2, to expand $$ f(x)$$ by the powers of $$ x$$, getting

$$displaystyle f(x) = frac{1}{(p!-!1)!}[(-1)^{np}n!^px^{p-1}+A_1x^p+A_2x^p+1+ldots]$$

where $$ A_1,,A_2,,ldots$$ are integers, and to replace the powers $$ x^{p-1}$$, $$ x^p$$, $$ x^{p+1}$$, ... with the numbers $$ (p!-!1)!$$, $$ p!$$, $$ (p!+!1)!$$, ... We then get the expression

$$displaystyle F(0) = frac{1}{(p!-!1)!}[(-1)^{np}n!^p(p!-!1)!+A_1p! +A_2(p!+!1)!+ldots] =
(-1)^{np}n!^p+pK_0,$$

in which $$ K_0$$ is an integer.

We now set for the prime $$ p$$ the condition $$ p > n$$. Then, $$n!$$ is not divisible by $$p$$, neither is the former addend $$ (-1)^{np}n!^p$$. On the other hand, the latter addend $$ pK_0$$ is divisible by $$p$$. Therefore:

($$ alpha$$) $$ F(0)$$ is a non-zero integer not divisible by $$ p$$.

For determining $$ F(1)$$, $$ F(2)$$, ..., $$F(n)$$ we expand the polynomial $$ f(x)$$ by the powers of $$ x!-!nu$$, putting $$ x := nu!+!(x!-!nu)$$. Because $$f(x)$$ contains the factor $$ (x!-!nu)^p$$, we obtain an expansion of the form

$$displaystyle f(x) = frac{1}{(p!-!1)!}[B_p(x!-!nu)^p+B_{p+1}(x!-!nu)^{p+1}+ldots],$$

where the $$ B_i$$'s are integers. Using the lemma 2 then gives the result

$$displaystyle F(nu) = frac{1}{(p!-!1)!}[p!B_p+(p!+!1)!B_{p+1}+ldots] = pK_{nu},$$

with $$ K_{nu}$$ a certain integer. Thus:

($$ beta$$) $$F(1)$$, $$ F(2)$$, ..., $$ F(n)$$ are integers all divisible by $$ p$$.

So, the left hand side of (4) is an integer having the form $$c_0F(0)+pK$$ with $$ K$$ an integer. The factor $$ F(0)$$ of the first addend is by ($$ alpha$$) indivisible by $$ p$$. If we set for the prime $$ p$$ a new requirement $$ p >mid c_0mid$$, then also the factor $$ c_0$$ is indivisible by $$ p$$, and thus likewise the whole addend $$ c_0F(0)$$. We conclude that the sum is not divisible by $$ p$$ and therefore:

($$ gamma$$)   If $$ p$$ in (5) is a prime number greater than $$ n$$ and $$ mid c_0mid$$, then the left side of (4) is a non-zero integer.

We then examine the right hand side of (4). Because the numbers $$ xi_1$$, ..., $$ xi_n$$ all are positive (cf. (3)), so the exponential factors $$e^{1-xi_1}$$, ..., $$e^{n-xi_n}$$ all are $$ < e^n$$. If $$ 0 < x < n$$, then in the polynomial (5) the factors $$ x$$, $$ x!-!1$$, ..., $$x!-!n$$ all have the absolute value less than $$ n$$ and thus

$$displaystyle mid f(x)mid < frac{1}{(p!-!1)!}n^{p-1}(n^n)^p = n^ncdotfrac{(n^{n+1})^{p-1}}{(p!-!1)!}.$$

Because $$ xi_1$$, ..., $$xi_n$$ all are between 0 and $$ n$$ (cf. (3)), we especially have

$$displaystyle mid f(xi_{nu})mid < n^ncdotfrac{(n^{n+1})^{p-1}}{(p!-!1)!} quadforall ,nu = 1,,2,,ldots,,n.$$

If we denote by $$ c$$ the greatest of the numbers $$mid c_0mid$$, $$mid c_1mid$$, ..., $$ mid c_nmid$$, then the right hand side of (4) has the absolute value less than

$$displaystyle (1!+!2!+!ldots!+!n)ce^nn^ncdotfrac{(n^{n+1})^{p-1}}{(p!-!1)!} = frac{n(n!+!1)}{2}c(en)^ncdotfrac{(n^{n+1})^{p-1}}{(p!-!1)!}.$$

But the limit of $$ frac{(n^{n+1})^{p-1}}{(p!-!1)!}$$ is 0 as $$ptoinfty$$, and therefore the above expression is less than 1 as soon as $$ p$$ exeeds some number $$ p_0$$.

If we determine the polynomial $$ f(x)$$ from the equation (5) such that the prime $$ p$$ is greater than the greatest of the numbers $$ n$$, $$mid c_0mid$$ and $$ p_0$$ (which is possible since there are infinitely many prime numbers), then the left side of (4) is a non-zero integer and thus $$geq 1$$, whereas the right side having the absolute value $$ < 1$$. The contradiction proves that the theorem is right.