Norm Induced Topologies on Finite Dimensional Real Vector Spaces

by Bishal Deb - Sunday, May 10, 2015

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In this article we show that the topology induced by a norm on a finite dimensional real vector space is an intrinsic topology irrespective of the norm. It is equivalent for all norms.

Let V be a finite dimensional vector space over $$mathbb{R}$$ with dimension n. Let $$mathscr{B}={x_1,ldots,x_n}$$ be a basis for V.

An example of a norm would be $$N(x)_{mathscr{B}}=sqrt{a_1^2+ldots+ a_n^2}$$ where $$x=a_1x_1+ldots+a_nx_n$$ for some $$a_1,ldots,a_ninmathbb{R}$$.

The above is the norm that we generally work with and know as the absolute value or modulus of a real or complex number. But throughout this article we use the following norm : $$N^{backprime}(x)_{mathscr{B}}=displaystyle{sup_i}{x_1,ldots,x_n}$$. We write $$N^{backprime}(x)_{mathscr{B}}=||x||_{infty,mathscr{B}}$$. These norms are known as $$L^{infty}$$ norms and depend on the choice of basis.

To prove our claim we construct a topology using this norm. Then we show that the topology constructed is independent of the choice of basis. And finally we show that any other norm would induce the same topology.

Construction of the Topology

Let $$x_{circ}in V$$ and for any $$epsilon >0$$ we define an open box around $$x_{circ}$$ to be the set of all $$x$$ such that $$||x-x_{circ}||_{infty,mathscr{B}}<epsilon$$.

Hence in $$mathbb{R}$$ it is an open interval; in $$mathbb{R}^2$$ it is the open rectangle and similarly in higher dimensions it is the open hyper rectangle.

We take the family of open boxes around all vectors to be the basis of the topology induced by the $$L^{infty}$$ norm with respect to basis $$mathscr{B}$$.

Note that any norm is a continous function with this definition.

Equivalence of $$L^{infty}$$ norms

Let $$mathscr{B}^{backprime}={x^{backprime}_1,ldots,x^{backprime}_n}$$ be some other basis on V. We show that the norms $$N^{backprime}_{mathscr{B}^{backprime}}$$ and $$N^{backprime}_{mathscr{B}}$$ induce the same topology.

Let $$N : Vrightarrow mathbb{R}^+$$ be some norm. Then,

$N(x)=N(a_1x_1+ldots+a_nx_n)leq displaystyle{sup_i} (|a_i|)N(x_1ldots+x_n) = ||x||_{infty,mathscr{B}}N(x_1+ldots+x_n)$ (1)

where $x=a_1x_1+ldots+a_nx_n$ as per basis $mathscr{B}$.

But $N(x_1+ldots+x_n)$ is a constant depending on the basis and the norm and independent of x.

By putting $N=N^{backprime}_{mathscr{B^{backprime}}}$ in (1) we have,

$N^{backprime}_{mathscr{B^{backprime}}}(x)=N^{backprime}_{mathscr{B^{backprime}}}(a_1x_1+ldots+a_nx_n)leq displaystyle{sup_i} (|a_i|)N^{backprime}_{mathscr{B^{backprime}}}(x_1ldots+x_n) = c_1||x||_{infty,mathscr{B}}$ (2)
where c_1 is a constant.

Similarly exchanging $N^{backprime}_{mathscr{B}}$ with $N^{backprime}_{mathscr{B}^{backprime}}$ and plugging $N=N^{backprime}_{mathscr{B}}$ in (1) we have a similar inequality,

$N^{backprime}_{mathscr{B}}(x)=N^{backprime}_{mathscr{B}}(a_1x_1+ldots+a_nx_n)leq displaystyle{sup_i} (|a_i|)N^{backprime}_{mathscr{B}}(x_1ldots+x_n) = frac{1}{c_2}||x||_{infty,mathscr{B^{backprime}}}$ (3)
where c_2 is a constant and $x=a^{backprime}_1x^{backprime}_1+ldots+a^{backprime}_nx^{backprime}_n$ as per basis $mathscr{B^{backprime}}$.

Combining the two inequalities (2) and (3) we have
$$c_2||x||_{infty,mathscr{B}}leq ||x||_{infty,mathscr{B^backprime}}leq c_1||x||_{infty,mathscr{B}}$$

Hence as both the norms are comparable the topology induced is independent of the basis that we choose for such norms.

As it does not depend upon the basis we shall hereafter use $N^{backprime}$ and $||||_{infty}$ for convenience of notation.

Equivalence of the L^{infty} norm induced topology with any other norm induced topology
We apply a similar method like in the above section, i.e we show that any norm is comparable with the norm $N^{backprime}$. Let $N_1:Vrightarrow mathbb{R}^+$ be some norm. We replace N in equation (1) by N_1 to get one side of the desired inequality.

We now get the inequality on the other side.

We consider the set $A={x|text{ } ||x||_{infty}=1, xin V}$. Clearly A is compact. As any norm is continuous so is N_1. Hence $N_1(A)$ is also compact and its minimum exists in A. Also the minimum is a non-zero constant. From this the other side of the inequality easily follows.

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