Problem Set prepared by B. J. Venkatachala for Olympiad Orientation Programme-2014, North-East Regions

by Gonit Sora - Tuesday, August 18, 2015

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1) A triangle has sides 13, 20, 21. Is there an altitude having integral length?

2) What is the minimum number of years needed for the total number of months in them is a number containing only the digits 0 and 1?

3) Suppose a, b are integers such that 9 divides $$a^2+ab+b^2.$$ Prove that 3 divides both a and b.

4) Suppose x and y are real numbers such that $$(x+\sqrt{x^2+1})(y+\sqrt{y^2+1})=1.$$ find x+y.

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5) Solve the system for positive real x, y : $$x^2+y=7, x+y^2=11.$$

6) Suppose p and p^2+2 are primes. Prove that p^3+2 is also a prime.

7) Prove that is 2n+1 and 3n+1 are square numbers for some positive integers n, then 5n+3 can't be a prime number.

8) Show that $65^{64}+64$ is a composite number.

9) Four different digits are chosen, and all possible positive four-digit numbers of distinct digits are

constructed out of them. The sum of these four-digit numbers is found to be 186648. What me be the four digits used?

10) Solve the simultaneous equations

$$x-xy+y=1, x^2+y^2=17.$$

11) Given eight 3-digit numbers, from all possible 6-digit numbers by writing two 3-digit numbers side-by-side. Prove that among these 6-digit numbers, there is always a number divisible by 7.

12) Find all pairs of positive integers (m, n) such that $$|3^m-2^n|=1.$$

13) For any set of n integers, show that it contains a subset of whose elements are divisible by n.

14) Find all triples of natural numbers (a, b, c) such that the remainder after dividing the product of any two by the other is 1.

15) If a, b, c are real numbers such that a+b+c=0, prove that

$$frac{a^5+b^5+c^5}{5}= frac{a^3+b^3+c^3}{3}.frac{a^2+b^2+c^2}{2}.$$

16) Solve the equation:

$$16[x]^2+16{x}^2-24x=11.$$

17) Find the least positive integer having 30 positive divisors.

18) Let a and b be real numbers such that $$a^3-3a^2+5a-17=0$$ and $$b^3-3b^2+5b+11=0.$$ Find

a+b.

19) Is there a square number the sum of whose digits is 2015?

20) Find all numbers a, b such that $(x-1)^2$ divides $ax^4+bx^3+1.$

21) Suppose P(x) is a polynomial with integer coefficients such that P(0) and P(1) are both odd numbers, Prove that P(x)=0 has no integer root.

22) Let $p(x)=x^2+ax+b,$ where a, b are integers. Given an integer m. Prove that there exists an integer n such that $p(m)p(m+1)=p(n).$

23) Let P(x) be a cubic polynomial such that P(1)=1, P(2)=2, P(3)=3 and P(4)=5. Find P(6).

24) For any four positive real numbers $a_1,a_2,a_3,a_4,$ prove the inequality:

$$\frac{a_1}{a_1+a_2}+\frac{a_2}{a_2+a_3}+\frac{a_3}{a_3+a_4}+\frac{a_4}{a_4+a_5}\le\frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+\frac{a_3}{a_4+a_5}+\frac{a_4}{a_1+a_2}.$$

25) If a, b, c are positive real numbers, prove that

$$3(a+\sqrt{ab}+\sqrt[3]{abc})\le 4(a+b+c)).$$

26) How many zeros are there at the end of 1000!?

27) Suppose x, y, z are integers such that $x^2+y^2=z^2.$ Prove that 60 divides xyz.

28) Find all 5-term geometric progressions of positive integers whose sum is 211.

29) Find all arithmetic progressions of natural numbers such that for each n, the sum of the first n-terms of the progression is a perfect square.

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30) Consider the two squares lying inside a triangle ABC with $\angle A=90^{\circ}$ with their vertices on the sides of ABC: one square having its sides parallel to AB and AC, the other, having two sides parallel to the hypotenuse. Determine which of these two squares has greater area.

31) How many 5-digit numbers contain at least one 5?

32) Let a, b, c, d be four integers. Prove that (a-b)(a-c)(a-d)(b-c)(b-d)(c-d) is always divisible by 12.

33) Let N be a 16-digit positive integer. Show that we can find some consecutive digits of N such that the product of these digits is a square.

34) Let ABCD be a unit square and P be an interior point such that $\angle PAB=\angle PBA=15^{\circ}.$ Show that DPC is an equilateral triangle.

35) Let ABC be an isosceles triangle in which $\angle A=20^{\circ}.$ Let D be a point on AC such that AD=BC. Find $\angle ABD.$

36) Let ABC be an isosceles triangle in which $\angle A=100^{\circ}.$ Extend AB to D such that AD=BC. Find $\angle ADC.$

37) Let ABC be an isosceles triangle with AB=AC and $\angle A=20^{\circ}.$ Let D, E be points on

AB and AC respectively such that $$angle CBE=50^{circ}$$ and $$angle BCD=60^{circ}.$$ Determine $$angle EDC.$$

38) In a triangle ABC, the altitude, the angle bisector and the median from A divide $$angle A$$ in four equal parts. Find the angles of ABC.

39) In an equilateral triangle ABC, there is a point P which is at a distance 3, 4, 5 from the three vertices respectively. What is the area of the triangle?

40) In a square ABCD, there is a point P such that PA=3, PB=7 and PD=5. What is the area of ABCD?

41) Let $$x_1,x_2$$ be the roots of $$x^2+ax+bc=0$$ and $$x_2,x_3$$ be those of $$x^2+bx+ac=0.$$ Suppose $$acne bc.$$ Prove that $$x_1,x_3$$ are the roots of $$x^2+cx+ab=0.$$

42) The polynomial $$p(x)=ax^3+bx^2+cx+d$$ has integer coefficients a, b, c, d with ad odd and bc even. Prove that the equation p(x)=0 has at least one one-trivial root.

43) If a, b, c are the sides of a triangle of a triangle, prove that

$$frac{3}{2}lefrac{a}{b+c}+frac{b}{c+a}+frac{c}{a+b}<2.$$

44) Let a, b, c be the sides of a sides of a triangle such that

$$frac{bc}{b+c}+frac{ca}{c+a}+frac{ab}{a+b}=s,$$

Where s is the semi-perimeter of the triangle. Prove that the triangle is equilateral.

45) Let a, b, c, d be positive real numbers. Prove that

$$frac{a}{b+2c+3d}+frac{d}{c+2d+3a}+frac{c}{d+2a+3b}+frac{d}{a+2b+3c}gefrac{2}{3}.$$

46) Suppose n is a natural number such that 2n+1 and 3n+1 are both perfect squares. Prove that 40 divides n.

47) Let ABC be a triangle in which AB<AC. Let D be the mid-point of the arc BC of the circumcircle of ABC containing A. Draw DE perpendicular to AC (with E on AC). Prove that AB+AE=BC.

48) Construct an equilateral triangle, only with ruler and compass, which has area equal to that of a given triangle.

49) Show that for each natural number n, the number of integer solutions (x,y) of the equation $$x^2+xy+y^2=n$$ is a multiple of 6.

50) For any $$nin N,$$ Let $$a_n$$ denote the number of positive integers whose digits are from the set {1,3,4} and the sum of the digits is n. prove that $$a_{2n}$$ is a perfect square for every $$nin N.$$

51) Solve $$2^t=3^x5^y+7^z$$ in positive integers.

52) Let n be a positive integers such that 2n+1 and 3n+a are perfect squares. Prove that 5n+3 is a composite integers.

53) Let S denote the set of all integers which can be expressed in the form $$a^3+b^3+c^3-3abc,$$ where a, b, c are integers. Prove that S is closed under multiplication.

54) Let a, b, c, d be positive integers such that both $$frac{a}{b}+frac{b}{c}+frac{c}{a}$$ and $$frac{a}{c}+frac{b}{a}+frac{c}{b}$$ are integers. Prove that a=b=c.

55) Positive integers a, b, c are such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1.$ Prove that

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{41}{42}.$

56) Find all integers x, y, z such that $x^3+2y^3=4z^3.$

57) Find the largest power of 3 that divides $10^k-1,$ where k is any positive integer.

58) Find the sum $\sum^{100}_{k=1}\frac{k}{k^4+k^2+1}.$

59) Find all ordered pairs (p,q) of prime numbers such that pq divides $5^p+5^q.$

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60) Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1.$ Prove that $(a-1)(b-1)(c-1)\ge 8.$

61) Find the sum $\sum^{2014}_{k=1}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}.$

62) Around a circle are written all positive integers from 1 to N, $N\ge 2,$ in such a way that any two adjacent numbers have at least one common digit; for example, 12 and 26 can occur as adjacent numbers, but not 16 and 24. Find the least N for which this is possible.

63) The length of the sides of a quadrilateral are positive integers. It is known that the sum of any three numbers is divisible by the fourth-one. Prove that two sides of the quadrilateral are equal.

64) Prove that $n^{12}+64$ has at least 4 distinct factors (other than 1 and itself), for any n>1.

65) Suppose a, b, c, d are integers such that a+b+c+d=0. Prove that $$2a^4+2b^4+2c^4+2d^4+8abcd$$ is a perfect square.

66) Solve the simultaneous equations:

$$asqrt{a}+b asqrt{b}=183, asqrt{b}+bsqrt{a}=182.$$
