Problem Set prepared by C. R. Pranesachar for Olympiad Orientation Programme-2014, North-East Regions

by Gonit Sora - Tuesday, August 18, 2015

http://gonitsora.com/problem-set-prepared-by-c-r-pranesachar-for-olympiad-orientation-programme-2014-north-east-regions/

1) Let P be an interior point of an equilateral triangle ABC.

(a) Calculate the side of the triangle if the distances of P from the sides BC, CA, AB are given as x, y, z.

(b) Calculate the side of the triangle if the distances of P from the vertices A, B, C are given as x, y, z.

2)

(a) Prove that every positive integer n can written as $latex \pm 1 \pm 2 \pm 3 \pm \ldots \pm k$, (for a suitable choice of signs + and -) for some k.

(b) Prove that every positive integer n can written as $latex \pm 1^2\pm 2^2\pm 3^2\pm\ldots\pm k^2$, (for a suitable choice of signs + and -) for some k.

3) In the figure O is an interior point of a triangle ABC, and the lines KL, MN, PQ passing through O are respectively parallel to BC, CA, AB with Q, M on BC; L, P on CA; and N, K on AB. If $latex [AKL]=S_1, [BMN]=S_2, [CPQ]=S_3, [ABC]=S,$ find $latex S$ in terms of $latex S_1,S_2,S_3.$ Here [] denotes area. Also f in $latex S$ in terms of $latex [OQM]=T_1, [OLP]=T_2, [ONK]=T_3$ as well as in terms of $latex [ANOP]=X, [BQOK]=Y, [CLOM]=Z.$

4) Two villages A and B are on the same side of a straight road. At which point should a bus shelter be put up on the road so that the cost of laying roads from the villages to the bus shelter is a minimum.

5) What is the minimum value of $latex sqrt{x^2+4}+sqrt{x^2-6x+34}$ as x varies over all real values?

6) Let ABCD be an isosceles trapezium that has the property that the lengths of its sides and diamonds

(six in all) from a two-element set {a,b}. If a>b, determine a/b.

7) In the star-shaped figure given here find the sum of the 5 corner angles that are marked.

8)

(a) There are six points in a plane no three of which are collinear. Every pair of points is joined by a blue or red line. Show that there always results a monochromatic triangle (coloring is arbitrary). The conclusion is not true if we start with five points.

(b) There are seventeen points in a plane no three of which are collinear. Every pair of points is joined by a blue, red or green line. Show that there always results a monochromatic triangle (coloring is arbitrary).

9)

(a) If we color all the points of a line by 2 colors, then there exist three distinct points A, B, C such that they have the same color and AB=AC.

(b) If we color all the points of a plane by 2 color, then there exist three distinct points A, B, C such that they have the same color and AB=BC=CA.

(c) If we color all the points of a plane by 3 colors, then there are two points A, B such that they both have the same color.

10) Solve the system for real x, y, z:

$x+[y]+\{z\}=13.0$

$y+[z]+\{x\}=13.5$

$z+[x]+\{y\}=12$

11) Prove that $latex sqrt{2sqrt{7}+4}=sqrt{2sqrt{7}+1}+sqrt{2sqrt{7}-5}.$

12) If a, b, c are positive real numbers and all the roots of the cubic equation $x^3-ax^2+bx-c=0$ are real, then must the roots be all positive?

13)

(a) The medians of a triangle divide it into 6 smaller triangles of equal area. Prove.

(b) Can the medians of a triangle always form the sides of a triangle?

Image Source : Shutterstock

14) In trapezium ABCD, AB is parallel to CD. AC, BD intersect at O. If [AOB]=x, [COD]=y, find [ABCD].

15) There are 3 cans of measures 10 litters, 7 litters and 3 litters. The 10 litter-can is full of oil and other two are empty. Without using any other measure divide the oil quantity into 2 equal halves.

16)

(a) Let n be an odd integer, positive or negative. Prove that n can be expressed as a difference of two squares.

(b) Let n be any integer. Prove that n can be expressed in the form $n=a^2+b^2-c^2$ for suitable positive integers a, b, c.

17)

(a) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots+\frac{1}{99}-\frac{1}{100}=\frac{1}{51}+\frac{1}{52}+\frac{1}{53}+\ldots+\frac{1}{100};$

(b) $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{97.100}=\frac{33}{100};$

(c) $\frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\ldots.\frac{99}{100}<\frac{1}{\sqrt{101}}.$

18) For which integers n, does the number n!=1.2.3....n end in

(a) exactly 4 zeros?

(b) exactly 5 zeros?

(c) 100 zeros?

19) In how many ways can a person climb a flight of 12 steps, if one step or two can be climbed at a time?

20)

(a) Can the altitudes of a triangle be 4, 7, 10?

(b) Can the sides of a triangle be each > 1000 cm and its area be equal to 1 $latex cm^2$?

(c) Can the sides of a triangle be each < 1 cm and its circumradius be 1000 cm?

21) If A is a positive integer such that 2A is a square, 3A is a cube, determine the least possible of A. What if further, 5A is a fifth power?

22) Among n given positive integers, there are some whose sum is divisible by n. Prove.

23) Among 5 given integers, there are three whose sum is divisible by 3. Prove.

24) In the $latex 5\times 5$ grid of 25 pints given here, how many triples of points can be chosen to form the vertices of a triangle?

25) Can the altitudes of a triangle always form the sides of a triangle?

Image Source : Shutterstock

26)

(a) Find all integers-sided rectangles whose area equals the perimeter.

(b) Find all integer-sided cuboids whose volume equals the total surface area.

27) Two sides of an integer-sided triangle are 8 and 17. How many possible values are there for the third side? Among all such triangles how many acute, right, obtuse triangles are there?

28) Using at most two straight cuts, dissect a triangle into smaller pieces and rearrange them to get a rectangle.

29) Let P be an interior point of a triangle ABC, with semi-perimeter s. Prove that $s < PA + PB + PC < 2s$.

30) Find the integer n such that

$latex frac{1}{1+sqrt{3}}+frac{1}{sqrt{2}+sqrt{3}}+frac{1}{sqrt{3}+sqrt{4}}+\ldots +frac{1}{sqrt{n}+sqrt{n+1}}=10.$