Simple concept Tough problem - 1

by Debashish Sharma - Friday, October 25, 2019

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Many tough mathematical problems require just basic mathematics for their solutions. This series of articles will discuss such problems on a regular basis. The problem for the first edition is taken from the Regional Mathematics Olympiad (RMO) 2019. It's question number 2 in the RMO paper. While there are surely several ways to solve, I found a solution that requires some basic results from school geometry.

Question : Let be a triangle with circumcircle and let be the centroid of triangle . Extend and to meet the circle again in respectively. Suppose and . Prove that and are equilateral triangles.

Well, the first step for any geometry problem is to draw a neatly labelled diagram. Do remember to keep the diagram as general as possible. In this problem, a very frequent mistake is to draw and as equilateral triangles. Never do that. If you do, the medians will look like altitudes and there will be a tendency to assume wrong things. So, let us have a look below at the diagram drawn from the given information :

Let be the medians of . Next we mark some angles. Let and . Immediately, from the well known result that angles on the same segment of a circle are equal, we get a few other angles. and . But , so that . This gives and so . Also, let so that . Then using we get so that . Next, let so that . Using , we get . We mark the angles in the following diagram.

Ah ! The diagram looks a bit decorated now ! Let's do some construction. Mark the line segments and . Also mark and . Mark the obvious angles that are formed. Most angles will have familiar names, thanks to the theorem stated earlier on angles on the same segment.

Thus we get . Hey wait, that means but they are alternate angles formed by the transversal intersecting the line segments and . Doesn't that mean ? Yes indeed !

Next, consider and . We have , (as AD is the median) and (vertically opposite angles). So, by congruency axiom, . Thus, . Here we are ! is actually a parallelogram. So its opposite angles will be equal. We have, so that i.e. .

Similarly, we can show is a parallelogram. So, . Thus, all angles of and are equal to . So, and are equilateral.

So, we see that almost half of the problem was solved just by drawing the properly labelled diagram. The concepts required were of parallel lines, congruence of triangles and a single theorem of circles ! Watch out for the next problem of SCTP series soon ! Stay tuned to GonitSora !!