Solving The Cubic

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People have been in an attempt to find roots of polynomials since a very long time. They may not have had a very concrete notion of a polynomial and may have been trying to solve some specific polynomials rather than solving some general polynomials. The quadratic equation was solved by Mesopotamians and Babylonians and Indians, etc. But the solution to the quadratic equation in its general form that we see today was done by Henry Heaton only in 1896.

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In many places in Europe people interpreted square terms as the area of a square with some unknown side length and similarly they interpreted the cube term as volume of a cube with some unknown side length. Hence the attempts at solving polynomials were mostly with geometric ideas and this mostly restricted them from trying to solve polynomials of degree greater than or equal to 4.

There was no consistent notation for expressing a polynomial. There was no equality symbol. The equal symbol '=' was first introduced in 1557 by a man named Robert Recorde in his book The Whetstone of Witte. Different groups of people used different ways to denote polynomials. The Indians used poetic verses, and in Europe different groups of people had their own notation.

The first known result to a solution to the cubic was by the Italian mathematician Sciopione del Ferro. It is known that he gave a solution for a weak version of the "depressed" cubic equation. His method was later rediscovered by Niccolò Fontana Tartaglia. This method was first written in Gerolamo Cardano's *Ars Magna*(The Great Art).

Cardano learned of the method from Tartaglia. It so happened that one of Tartaglia's students challenged Tartaglia to solve the following equation :

$$x^3+cx=d$$

(1)

where c and d are both positive.

It seems that Cardano assumed that c and d are positive integers or fractions only, though he never mentions this in his book. Tartaglia, in a letter to Cardano, gives him a rule for computing a root for an equation of the above type in an Italian verse.

In order to find a solution it will be sufficient to find u, v such that

$$u-v=d$$

and $$uv = (dfrac{c}{3})^3$$

Then a solution will be $$sqrt[3]{u}-sqrt[3]{v}$$. This can be easily checked by plugging in place of x, $sqrt[3]{u}-sqrt[3]{v}$ in (1).

Hence the problem boils down to finding u,v but it can be seen that u,v can be found given c and d. Hence we get the following root

$$sqrt[3]{sqrt{frac{d^2}{4}+frac{c^3}{27}}+frac{d}{2}}-sqrt[3]{sqrt{frac{d^2}{4}+frac{c^3}{27}}-frac{d}{2}}$$

But we notice that the fact that c,d are positive integers or fractions is not used in the method. Hence we may extend the method to find solutions to the equations of the type (1) where c and d are any numbers. This is called the depressed cubic equation.

If we have the general cubic

$$ax^3+bx^2+cx+d=0$$

(2)

we can make it a depressed equation. This we can do by taking $y=x-frac{b}{3a}$. Then the equation transforms into one which is depressed, i.e., one which doesn't have the x^2 term. Thus we get an equation which we can solve using the above method.

Note : The above method only provides one root to (2). As an exercise one can try extending this method to find all the three solutions of the general cubic with complex coefficients.

References :

1. van der Waerden, B. L., From Al Khwarizmi to Emmy Noether A History of Algebra
2. Stedall J., From Cardano's Great Art to Lagrange's reflections : Filling a Gap in the History of Algebra
3. Wikipedia
