State Level Mathematics Competition-2013-Category-IV : Assam Academy of Mathematics

by Gonit Sora - Wednesday, October 02, 2013

https://gonitsora.com/state-level-mathematics-competition-2013-category-iv-assam-academy-mathematics/

01 September 2013

(Class XI and XII)

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Marks: 10 X 10 = 100

Time: 1.30 pm to 4.30 pm

Answer the following ten questions

1. Prove that $latex 4(x_1^4+ x_2^4+\ldots + x_{14}^4)=7(x_1^3+ x_2^3+\ldots + x_{14}^3)$ has no solution in positive integers.(Hint: Suppose on the contrary $latex \sum_{k=1}^{14}(x_k^4-\frac{7}{4}x_k^3)=0.$ Also use $latex \sum (x_k-1)^4.$)
2. Find all non negative integers a, b, c, d, n that satisfy$latex a^2+b^2+c^2+d^2=7.4^n.$

   (Hint: Put n=0 and use $latex 2^2+1^2+1^2+1^2=7.$)
3. Let ABCD be convex quadrilateral. Suppose that the lines AB and CD intersect at E and the lines AD and BC intersect at F. Prove that the following statements are equivalent(i) a circle is inscribed in ABCD

   (ii) BE+BF=DE+DF

   (iii) AE-AF=CE-CF

   (Hint: Inscribe a circle in the quadrilateral ABCD to touch the quadrilateral at K, L, M and N.)

   Image Source : Shutterstock
4. A and B are two points situated on the same side of a line XY. Find the position of a point M on the line such that the sum AM+MB is minimal.(Hint: Suppose B´ is the reflection of B across the line XY. M is the point of intersection of AB´ and XY.)
5. Let a, b, c be non-zero real numbers such that $latex a+b+c\neq 0$ and $latex \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}.$Prove that $latex \frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}.$

OR

Prove that is p is a prime the $latex \sqrt{p}$ is an irrational number.

6. Let a, b, c be positive numbers such that $latex abc=1.$ Prove that$latex \frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$

   (Hint: Use inequality $latex \frac{a^2}{x}+\frac{b^2}{y}\geq\frac{(a+b)^2}{x+y}$ in $latex \frac{\frac{1}{a^2}}{ab+ac}+\frac{\frac{1}{b^2}}{ab+bc}+\frac{\frac{1}{c^2}}{ac+bc}.$)
7. Find all functions $latex f:Q \rightarrow Q$ such that$latex f(x+y)+f(x-y)=2f(x)+2f(y),$ for all rationals x, y.

   OR

   The sum of two integers is 52 and their L.C.M. is 168. Find the numbers.
8. Show that there does not exist a function $latex f:N\rightarrow N$ which satisfy(a) $latex f(2)=3$

   (b) $latex f(mn)=f(m)f(n)$ for all $latex m,n$ in $latex N.$

   (c) $latex f(m)<f(n)$ whenever $latex m<n.$
9. If $latex a_1\leq a_3\leq\ldots \leq a_n$ be n real numbers such that $latex \sum_{j=1}^na_j=0.$Show that $latex na_1a_2+\sum_{j=1}^na_j^2\leq 0.$

   OR

   Prove that in a $latex \triangle ABC , \angle A=2 \angle B$ if and only if $latex a^2=b(b+c).$
10. The number of class of 27 pupils each goes swimming on some of the days from Monday to Friday in a certain week. If each pupil goes atleast twice, show that there must be two pupils who go swimming on exactly the same days.

    OR

    Let ABC be an acute angled triangle; AD be the bisector of $latex \angle BAC$ with D on BC; BE be the altitude from B on AC. Show that $latex \angle CED>45^{\circ}.$
