The Ubiquitous Pi

by Manjil Saikia - Sunday, May 15, 2011

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Let’s begin by Quoting William Schaff: ‘probably no symbol in mathematics has evoked as much mystery, romanticism, misconception and human interest as the number $$\pi$$....’

Geometry and trigonometry

For any circle with radius r and diameter d (= 2r), the circumference is $$\pi d$$ and the area is $$\pi r^2$$. Further, $$\pi$$ appears in formulas for areas and volumes of many other geometrical shapes based on circles, such as ellipses, spheres, cones, and tori.[52] Accordingly, $$\pi$$ appears in definite integrals that describe circumference, area or volume of shapes generated by circles. In the basic case, half the area of the unit disk is given by:

$$\int_{-1}^{1} \sqrt{1-x^2},dx = \frac{\pi}{2}$$

and

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}},dx = \pi$$

gives half the circumference of the unit circle. More complicated shapes can be integrated as solids of revolution.

From the unit-circle definition of the trigonometric functions also follows that the sine and cosine have period 2$$\pi$$. That is, for all x and integers n, sin(x) = sin(x + 2$$\pi$$n) and cos(x) = cos(x + 2$$\pi$$n). Because sin(0) = 0, sin(2$$\pi$$n) = 0 for all integers n. Also, the angle measure of 180° is equal to $$\pi$$ radians. In other words, 1° = ($$\pi$$/180) radians.

In modern mathematics, $$\pi$$ is often defined using trigonometric functions, for example as the smallest positive x for which sin x = 0, to avoid unnecessary dependence on the subtleties of Euclidean geometry and integration. Equivalently, $$\pi$$ can be defined using the inverse trigonometric functions, for example as $$\pi$$ = 2 arccos(0) or $$\pi$$ = 4 arctan(1). Expanding inverse trigonometric functions as power series is the easiest way to derive infinite series for $$\pi$$.

Complex numbers and calculus

The frequent appearance of $$\pi$$ in complex analysis can be related to the behavior of the exponential function of a complex variable, described by Euler's formula

$$e^{ivarphi} = \cos varphi + isin varphi$$

where i is the imaginary unit satisfying i2 = $$\pi$$1 and e $$\approx$$ 2.71828 is Euler's number. This formula implies
that imaginary powers of $e$ describe rotations on the unit circle in the complex plane; these rotations have a period of $360^\circ = 2\pi$. In particular, the $180^\circ$ rotation $\theta = \pi$ results in the remarkable Euler's identity

$$e^{i \pi} = -1.$$ 

There are $n$ different $n$-th roots of unity

$$e^{2 \pi i k/n} \quad (k = 0, 1, 2, \ldots, n - 1).$$

The Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$ 

A consequence is that the gamma function of a half-integer is a rational multiple of $\pi$.

**Physics**

Although not a physical constant, $\pi$ appears routinely in equations describing fundamental principles of the Universe, due in no small part to its relationship to the nature of the circle and, correspondingly, spherical coordinate systems. Using units such as Planck units can sometimes eliminate $\pi$ from formulae.

- The cosmological constant:

$$\Lambda = \frac{8\pi G}{3c^2}\rho$$

- Heisenberg's uncertainty principle, which shows that the uncertainty in the measurement of a particle's position ($\Delta x$) and momentum ($\Delta p$) cannot both be arbitrarily small at the same time:

$$\Delta x, \Delta p \geq \frac{\hbar}{4\pi}$$

- Einstein's field equation of general relativity:

$$R_{ik} - \frac{g_{ik} R}{2} + \Lambda g_{ik} = \frac{8\pi G}{c^4} T_{ik}$$

- Coulomb's law for the electric force, describing the force between two electric charges ($q_1$ and $q_2$) separated by distance $r$:

$$F = \frac{|q_1q_2|}{4\pi \varepsilon_0 r^2}$$

- Magnetic permeability of free space:

$$\mu_0 = 4\pi \times 10^{-7} \text{N/A}^2,$$

- Kepler's third law constant, relating the orbital period ($P$) and the semimajor axis ($a$) to the masses ($M$ and $m$) of two co-orbiting bodies:
$$\frac{P^2}{a^3} = \frac{(2\pi)^2}{G (M+m)}$$

**Probability and statistics**

In probability and statistics, there are many distributions whose formulas contain \(\pi\), including:

- The probability density function for the normal distribution with mean \(\mu\) and standard deviation \(\sigma\), due to the Gaussian integral:

$$f(x) = \frac{1}{\sigma \sqrt{2\pi} }, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- The probability density function for the (standard) Cauchy distribution:

$$f(x) = \frac{1}{\pi (1 + x^2)}$$.

Note that since $$\int_{-\infty}^{\infty} f(x),dx = 1$$ for any probability density function \(f(x)\), the above formulas can be used to produce other integral formulas for \(\pi\).

Buffon's needle problem is sometimes quoted as an empirical approximation of \(\pi\) in "popular mathematics" works. Consider dropping a needle of length \(L\) repeatedly on a surface containing parallel lines drawn \(S\) units apart (with \(S > L\)). If the needle is dropped \(n\) times and \(x\) of those times it comes to rest crossing a line \((x > 0)\), then one may approximate \(\pi\) using the Monte Carlo method:

$$\pi \approx \frac{2nL}{xS}.$$ 

Though this result is mathematically impeccable, it cannot be used to determine more than very few digits of \(\pi\) by experiment. Reliably getting just three digits (including the initial "3") right requires millions of throws, and the number of throws grows exponentially with the number of digits desired. Furthermore, any error in the measurement of the lengths \(L\) and \(S\) will transfer directly to an error in the approximated \(\pi\). For example, a difference of a single atom in the length of a 10-centimeter needle would show up around the 9th digit of the result. In practice, uncertainties in determining whether the needle actually crosses a line when it appears to exactly touch it will limit the attainable accuracy to much less than 9 digits.

**Pi in popular culture**

A whimsical "Pi plate"

Probably because of the simplicity of its definition, the concept of \(\pi\) and, especially its decimal expression, have become entrenched in popular culture to a degree far greater than almost any other mathematical construct. It is, perhaps, the most common ground between mathematicians and non-mathematicians. Reports on the latest, most-precise calculation of \(\pi\) (and related stunts) are common news items. [Pi Day (March 14, from 3.14) is observed in many schools. [At least one cheer at the Massachusetts Institute of Technology includes "3.14159!" The Signals catalogue offers a "Pi plate": a pie dish with both "\(\pi\)" and the decimal expression appearing on it.
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