The Vocabulary of Mathematical Analysis

by Manjil Saikia - Friday, November 09, 2012

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1. Introduction

The English word “vocabulary” refers to the “list of words” used in communicating one’s thought (or knowledge) to the others, and the English word “analysis” refers to the “resolution into simple elements”. Hence, according to the English dictionary, the “vocabulary of analysis” should refer to the **list of words/phases** used in the **resolution into simple elements** of the **problems of our concern** viz. **the mathematical problems**. The difficulties that creep in here are the following:

A. The mathematical meaning of the words/phrases used are sometimes different from their dictionary meaning, e.g. in “a number divisible by 4 is even”, **a** has been used not in the singular sense, but to imply ‘every’.

B. The same word/phrase is sometimes used to convey different implications, e.g. “point” of a set, “point” of the Euclidean plane.

C. The simple elements into which the resolutions are made are sometimes not defined beforehand with adequate care, they are simply left to intuition, e.g. ‘points’ and ‘lines’ in Euclidean plane; ‘variables’ in mathematical analysis.

In particular, in mathematical analysis, one needs to pay special attention to these difficulties, in order to make it intelligible and interesting to the beginners.

2. Some of the words constituting the vocabulary of mathematical analysis

The words that from the basis of vocabulary of mathematical analysis are “variable”, “function”, “sequence”, “infinity”. Each of the words requires a good introduction and a carefully laid foundation to which we now turn.

2.1 Variable

‘Variable’ is a word used very frequently and, probably, most freely in mathematical analysis, but is seldom honored with a proper introduction. Historically, the notion of ‘Variable magnitude’ was first applied by the French mathematician Rene’ Descartes (1596-1650 A.D.) in his book “Geometry”, published in 1637. Thousand years before Descartes, the Indian mathematician Brahmagupta (598-630? A.D.) used “antar” (differences/variations) to find the sines of intermediate angles in his book “Khandakhadyaka”. For a formal definition of a variable we had to wait till the nineteenth century, and it came from the German mathematician Karl Weierstrass (1815-1897):

A **“variable”** is represented by a letter, say x, that can designate any number of a prescribed set, say S.

The set S is said to be the domain of variation of the variable x; x is said to vary on S.

An **"independent variable"** is a variable whose variation on its domain is not governed by any rule whatsoever; on the contrary, a **dependent variable** is one whose variation is governed by some rule/s; e.g. the distance travelled by a car is a variable, dependent on the variable that designates time, whose variation is of course independent.

A **continuous variable** is a variable whose domain of variation is an interval; e.g. in the previous example time is a continuous variable. In contrast, a **discrete variable** is one whose domain of variation consists of some discrete real numbers.

A **parameter** is also a variable, whose variation determines different members or parts of a problem; e.g. in the problem of finding the distance covered by a cycle, a motor car, a train and an aeroplane, is a parameter.

2.2 Function

The English word "function" may lead us to some auditorium for visual entertainment or it may refer to the activities of some person or some tool. These implications of the word **function** do not obviously tally with its mathematical implication.

The ideas of variability and functionality, although prevailing all through the development of mathematics, were fist crystallized by Leonard Euler (1707-1783 A.D).

A **function** is a trio that consist of two sets D, R and a rule f, which assigns a **unique** member of R to each member of D. Symbolically we write '$$f:Drightarrow R$$' or '$$Drightarrow^f R$$'.

The set D is known as the **domain** of the function and the set R is known as the **range** of the function.

The member of R that corresponds to the member x of D under the rule f is denoted by f(x). We also write $$xrightarrow f(x).$$

N.B.— WRONG PHRASE: **"Let f(x) be a function"**

Although this phrase occurs frequently in the book of analysis, it should be noted that "f(x)" cannot be a function. In fact, "f(x)" is a member of the range R of the function '$$f:Drightarrow R$$', that corresponds to the member of the domain D under the rule f.

In some cases the rule f plays the dominant role in the function '$$f:Drightarrow R$$' in the sense that, given the rule of f, its domain D of action can be easily guessed, whence the range R is automatically determined; e.g. $$f(x)=+sqrt{1-x^2};$$ here D=[-1,1] and R=[0,1]. Hence, in case of this type it is enough to prescribe the rule f of the function without mentioning its domain or range. This probability is responsible for the introduction of the wrong phrase 'let f(x) be a function'.

RECTIFICATION: Instead of using the wrong phrase given above, one may use the phrase "let f be a function", or better perhaps, "let f(.) be a function".

Depending on the domain D and the range R, a function is categorized as follows:

(i) If $$Dsubset R$$ (the set of real numbers), then '$$f:Drightarrow R$$' is known as a **function of a real variable**.

(ii) If $$Rsubset R,$$ then '$$f:Drightarrow R$$' is a **real valued function**.

(iii) If R={c}, the singleton set containing only the real number c, '$$f:Drightarrow R$$' is called a **constant function**.

N.B.— A word of clarification is probably necessary here; a constant function maps every member of its domain to one and the same elements, viz. c in R. The element c, considered only as an element of R, is sometimes referred to as **absolute constant**, in order to avoid confusion with constant function.

(iv) If D=N (the set of natural numbers), the function $$f:Nrightarrow R$$ has been given a special name, viz. **sequence**.

A sequence is therefore a function whose domain is the set of natural numbers.

It is customary to denote the image of the positive integer n under the function $$f:Nrightarrow R,$$ i.e. f(n), by $$x_n.$$ Then clearly $$R={x_1,x_2,x_3,dots ,x_n,dots}.$$ Obviously R is the important part of any sequence, as the domain of definition of each sequence is N. Hence, it is enough to know the range $$R={x_n:nin N}$$ of a sequence. For this reason perhaps the use of the phrase “let $${x_n}$$ be a sequence” has become so common, knowing fully well that in no way can the symbol $${x_n}$$ stand for a sequence which is a function.

RECTIFICATION: The two objections against the notation $${x_n}$$ are the following:

(a) The symbol $${x_n}$$ stands for the singleton set containing only the element $$x_n;$$ hence it is not very judicial to the same symbol for a sequence.

(b) Even if $${x_n}$$ is supposed to stand for the set $${x_n:nin N},$$ it will only represent the range of the sequence and will thus hide its character as a function.

To overcome these objections, instead of the symbol $${x_n},$$ the symbol $${x_n}_n$$ may be used to represent the sequence which maps $$nin N$$ to $${x_n}.$$ Also $$“nrightarrow x_n”$$ may be used to denote the sequence.

2.3 Infinity

The idea of “infinitely small” and “infinitely large” arose naturally from successive subdivision and successive addition and were considered in as early as 4^{th} century B.C. by Aristotle (384-322 B.C.) and others. Since then the notion of **infinity** triggered off a lot of controversies which continued till the first half of the last century. While the distinctions between the finite and the infinite quantities were clearly appreciated by Galilei Galileo (1564-1642 A.D.), and the symbol '$$infty$$' (a horizontal or lying down 8) was used in connection with the notion of infinity by John Willis (1616-1703 A.D.), it required the genius of G. H. Hardy (1877-1947) to carefully formulate both the notion of infinity and the use of the symbol $\infty.$

The striking features of the symbol ∞ are:

(i) '∞' is just a symbol;

(ii) ∞ is not a number, ∞ is not belong to R.

(iii) each phrase containing the symbol is to be attributed a priori a ‘meaning’ with the help of a definition:

e.g.

(a) $(-\infty,\infty):=R;$ $[0,\infty):=\{x\in R:x\geq 0\}$

(b) "$n\rightarrow\infty$" or the phrase “n tends to infinity”.

The phrase refers to making the positive integer n larger and larger. It was Eudoxus of Cnidus (370 B.C.) who derived the area of a given circle by calculating the area A_n of a regular polygon P_n of n sides inscribed in the circle and then noting what happens to A_n on making n larger and larger.

The case of A_n as depicted above is only an example. Similar investigation can be carried out for any sequence $\{x_n\}_n:$ one asks what happens to x_n as n tends to infinity, or symbolically $\lim_{n\rightarrow\infty}x_n=?$

3. Some of the phrases used in mathematical analysis

Phrases like “however small”, “as small as we please”, “x approaches a” etc. are very common in “mathematical analysis”. Through their dictionary meanings are helpful, yet their mathematical meanings require special attention at least for the beginners. Unfortunately, this is what many of us tend to forget and thereby make “mathematical analysis” the most unintelligible.

The phrases “however small” and “as small as we please” express the notion of “infinitely small”; e.g. by making n large enough, one can make the difference between the area of a given circle and the area of the regular polygon of n sides inscribed in the circle “infinitely small” or “as small as we please” or less than any positive number “however small”.

In the phrase “x approaches a”, unlike the dictionary meaning of ‘approaches’, no motion is implies mathematically. The phrase requires the domain D of the variation of the real variable x to contain all the real numbers close to a. That is, there should exist some $\delta >0$ such that $(a-\delta ,a),(a,a+\delta)$ or both are in D. It should be noted that the point a may or may not belong to D. The phrase “x approaches a” implies that the differences $|x-a|$ can be made as small as we please.

4. Role of “mathematical analysis”

The main of mathematical analysis lies in investigating the nature of real valued functions of a real variable and the nature of the associated curves. In most cases it resolves the function under concern (near a given point a) into simple (known) elements; e.g. $$(x-a)^n,nin N$$ (Taylor's series), sin(cos)n(x-a) (Fourier series), etc. The problem of finding the curve whose family of tangents is known leads to the formal theory of "integration". The notion of variation permeates into each natural phenomenon allowing a fairly good model of the phenomenon in terms of differential equations. This is how mathematical analysis plays the central role in any investigation.

5. Conclusion:

Mathematical analysis is that branch of mathematics which claims to be associated with almost all investigations. As a result, mathematical analysis has become the core subject of study for all investigators. In view of this it is mandatory that the vocabulary of mathematical analysis be made as transparent and as lucid as possible. Both the authors of the books on mathematical analysis and the teachers of mathematical analysis are hereby urged to take extra hard care in their deliberations.

Author:- Prof. Jyoti Das.

(*Dr. (Mrs.) Jyoti Das* is Sir Ashutosh Birth Centenary Professor of Higher Mathematics and Ex-Head of the Department of Pure Mathematics, University of Calcutta.)

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